ON $\delta\theta$-SEQUENCES AND σ-PRODUCTS

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ABSTRACT. In this paper we shall obtain characterizations of $\delta\theta$-sequences and investigate $\delta\theta$-refinability-like properties of σ-products.

1. INTRODUCTION.

In 1967, J. M. W. Worrel [18] introduced the notion of θ-sequences and θ-refinable spaces and characterized θ-refinable spaces by using pointwise W-refining sequences. After that, H. J. K. Junnila [11, 12, 13] investigated θ-refinable spaces and characterized such spaces by using point star $\dot{F}$-refining sequences.

In this paper we investigate $\delta\theta$-sequences. We introduce the notions of pointwise countable W-refining sequences and point star $\dot{C}$ -refining sequences, and obtain a characterization of $\delta\theta$-refinability under an additional condition. Further we study $\delta\theta$-refinability-like properties of σ-products.

2. $\delta\theta$-SEQUENCES

Definition 1. A space X is called "$\delta\theta$-refinable" [3, p. 370] (resp. θ-refinable) if every open cover $\mathcal{G}$ of X has a $\delta\theta$-sequence (resp. θ-sequence) $(\mathcal{H}_n)_{n\in\mathbf{N}}$ of X such that each $\mathcal{H}_n$ is an open cover of X and a refinement of $\mathcal{G}$. Let us denote $\mathcal{H}_n \prec \mathcal{G}$ when $\mathcal{H}_n$ is a refinement of $\mathcal{G}$.

A sequence $(\mathcal{H}_n)_{n\in\mathbf{N}}$ of covers of X is called a "$\delta\theta$-sequence" (resp. θ-sequence) of X if for any $x \in X$ there is some $n_x \in \mathbf{N}$ such that $\mathrm{ord}(x, \mathcal{H}_{n_x}) \leq \omega$ (resp. $ord(x, \mathcal{H}_{n_x}) < \omega$) . Here $\mathrm{ord}(x, \mathcal{H}_{n_x}) = |\{H; x \in H \in \mathcal{H}_{n_x}\}|$ where ω denotes the first infinite ordinal and $|A|$ denotes the cardinal number of a set A.

Definition 2. ([12]). A family $\mathcal{L}$ of subsets of X is *interior preserving* if for each $\mathcal{K} \subset \mathcal{L}$, we have $\mathrm{Int}\bigcap\mathcal{K} = \cap\{\mathrm{Int}L | L \in \mathcal{K}\}$. Here IntL denotes the interior of L.

Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, define $\mathcal{U}_x = \{U | x \in U \in \mathcal{U}\}$.

Let $\mathcal{U}$ and $\mathcal{V}$ are open covers of X. $\mathcal{V}$ is called a pointwise W-refinement of $\mathcal{U}$ at x if there is a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{V}_x \prec \mathcal{U}'$. For every open cover $\mathcal{U}$ of X, let us put $\mathcal{U}^F = \{\bigcup\mathcal{U}' | \mathcal{U}' \subset \mathcal{U}, |\mathcal{U}'| < \omega\}$.

Concerning this, the following is known.

Theorem A ([12, Lemma 2.3]). Let $\mathcal{U}$ be an interior preserving open cover of X. Then the following are equivalent.

(1) There is an interior preserving open pointwise W-refinement $\mathcal{V}$ of $\mathcal{U}$.

(2) There is a closure preserving closed cover $\mathcal{F}$ of X such that $\mathcal{F} \prec \mathcal{U}^F$.

Now we shall introduce the notion of pointwise countable W-refinement and prove Theorems 1 and 2.

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Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ are open covers of X. $\mathcal{V}$ is called a pointwise countable W-refinement of $\mathcal{U}$ at x if there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{V}_x \prec \mathcal{U}'$. $\mathcal{V}$ is called a pointwise countable W-refinement of $\mathcal{U}$ if $\mathcal{V}$ is a pointwise countable W-refinement of $\mathcal{U}$ at x for every $x \in X$.

For every open cover $\mathcal{U}$ of X, let us put $\mathcal{U}^c = \{\bigcup \mathcal{U}' | \mathcal{U}' \subset \mathcal{U}, |\mathcal{U}'| \leq \omega\}$.

For each $x \in X$, we denote $\mathrm{st}(x,\mathcal{U}) = \bigcup\{U | x \in U \in \mathcal{U}\}$.

Theorem 1. *Let $\mathcal{U}$ be an interior preserving open cover of X. Then the following are equivalent.*
(1) There is an interior preserving open pointwise countable W-refinement $\mathcal{V}$ of $\mathcal{U}$.
(2) There is a closure preserving closed cover $\mathcal{F}$ of X such that $\mathcal{F} \prec \mathcal{U}^c$.

The proof of Theorem 1 is similar to that of Theorem 2 below.

Theorem 2. *Let $\mathcal{U}$ be an interior preserving open cover of X. Then the following are equivalent.*
(1) There is a sequence $(\mathcal{V}_n)_{n \in \mathbf{N}}$ of interior preserving open covers of X such that $\mathcal{V}_n \prec \mathcal{U}$ for each $n \in \mathbf{N}$ and for each $x \in X$, there is an n such that $\mathcal{V}_n$ is a pointwise countable W-refinement of $\mathcal{U}$ at x.
(2) There is a σ-closure preserving closed cover $\mathcal{F}$ of X such that $\mathcal{F} \prec \mathcal{U}^c$.

Proof. The basic idea of this proof is in the proof of [12, Lemma 2.3]. (2) $\Rightarrow$ (1). Let $\mathcal{F} = \cup_{n \in \mathbf{N}} \mathcal{F}_n$ be a closed cover of X such that $\mathcal{F} \prec \mathcal{U}^c$ and each $\mathcal{F}_n$ is a closure preserving family. For each $x \in X$, let $V_{n,x} = [\bigcap \mathcal{U}_x] \cap [X \setminus \bigcup(\mathcal{F}_n \setminus \mathcal{F}_x)]$. Then $V_{n,x}$ is open in X such that $x \in V_{n,x}$. Put $\mathcal{V}_n = \{V_{n,x} | x \in X\}$. Then $\mathcal{V}_n$ is an open cover of X such that $\mathcal{V}_n \prec \mathcal{U}$.

(i) $\mathcal{V}_n$ is interior preserving.

Proof. For each $A \subset X$, we have $\bigcap_{x \in A} V_{n,x} = [\bigcap \mathcal{U}_A] \cap [X \setminus \bigcup(\mathcal{F}_n \setminus \mathcal{F}_A)]$ where $\mathcal{U}_A = \{U | U \cap A \neq \emptyset\} = \cup_{x \in A} \mathcal{U}_x$ and $\mathcal{F}_A = \{F | F \cap A \neq \emptyset\} = \bigcup_{x \in A} \mathcal{F}_x$.

Since $\mathcal{U}$ is interior preserving, $\bigcap \mathcal{U}_A$ is open. Since $\mathcal{F}_n$ is closure preserving, $\bigcup(\mathcal{F}_n \setminus \mathcal{F}_A)$ is closed. Therefore $\bigcap_{x \in A} V_{n,x}$ is open.

(ii) For each $x \in X$, there exists an n such that $(\mathcal{V}_n)_x \prec \mathcal{U}'$ for some countable subfamily $\mathcal{U}'$ of $\mathcal{U}$.

Proof. For each $x \in X$, there exist an n and $F \in \mathcal{F}_n$ such that $x \in F$. Since $\mathcal{F}_n \prec \mathcal{U}^c$, there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $F \subset \bigcup \mathcal{U}'$.

Let $V \in (\mathcal{V}_n)_x$. Then $V = V_{n,y}$ for some $y \in X$. For each $F' \in \mathcal{F}_n \setminus \mathcal{F}_y$, since $x \in V_{n,y}, x \notin F'$. Since $x \in F, F \in \mathcal{F}_y$. Therefore $y \in F$. Hence $y \in \bigcup \mathcal{U}'$. Thus there is a $U \in \mathcal{U}'$ such that $y \in U$. Since $U \in \mathcal{U}_y, V_{n,y} \subset U$. Therefore $(\mathcal{V}_n)_x \prec \mathcal{U}'$.

(1) $\Rightarrow$ (2). Put $\mathcal{G} = \mathcal{U}^c$. For each $G \in \mathcal{G}$, let $F_{n,G} = \{x \in X | st(x, \mathcal{V}_n) \subset G\}$ and put $\mathcal{F}_n = \{F_{n,G} | G \in \mathcal{G}\}$. Then

(i) $F_{n,G}$ is closed in X.

Proof. Let $x \in X \setminus F_{n,G}$. Then $st(x, \mathcal{V}_n) \not\subset G$. Therefore there is $V \in \mathcal{V}_n$ such that $x \in V, V \not\subset G$. Put $O = \bigcap(\mathcal{V}_n)_x$ Then $x \in O$. Since $\mathcal{V}_n$ is interior preserving, O is open.

Let $y \in O$. Then $y \in V$. Since $V \not\subset G$, $\mathrm{st}(y, \mathcal{V}_n) \not\subset G$. Thus $y \notin F_{n,G}$. Hence $O \subset X \setminus F_{n,G}$.

(ii) $\mathcal{F} = \cup_{n \in \mathbf{N}} \mathcal{F}_n$ is a cover of X.

Proof. Let $x \in X$. There is an n such that $(\mathcal{V}_n)_x \prec \mathcal{U}'$ for some countable subfamily $\mathcal{U}'$ of $\mathcal{U}$. Put $G = \bigcup \mathcal{U}'$. Then $x \in F_{n,G}$.

(iii) $\mathcal{F}_n$ is closure preserving.

Proof. For each $\mathcal{G}' \subset \mathcal{G}$, put $F = \bigcup\{F_{n,G}|G \in \mathcal{G}'\}$. Then F is closed. To show this, let $x \in X \setminus F$. Then $x \notin F_{n,G}$ for each $G \in \mathcal{G}'$. Therefore there are $V_G \in \mathcal{V}_n$ such that $x \in V_G, V_G \nsubseteq G$. Put $V = \bigcap\{V_G|G \in \mathcal{G}'\}$. Then V is open, $x \in V$ and $V \cap F_{n,G} = \emptyset$ for each $G \in \mathcal{G}'$. Thus $V \cap F = \emptyset$. □

Worrel proved the following.
Theorem B([12, Proposition 1.4]). Let $\mathcal{U}$ be an open cover of X. Suppose there exists a sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of open refinements of $\mathcal{U}$ satisfying: for each $x \in X$ there is a sequence of integers $(\langle n,x\rangle)_{n\in\mathbf{N}}$ such that $\mathcal{U}_{\langle n+1,x\rangle}$ is a pointwise W-refinement of $\mathcal{U}_{\langle n,x\rangle}$ at x for each $n \in \mathbf{N}$. Then $\mathcal{U}$ has a θ-sequence of open refinements.

A family $\mathcal{L}$ of sets is called *monotone* if the partial order of set-inclusion is a linear order on L.

Concerning $\delta\theta$-sequences, we obtain the following.

Theorem 3. *Let $\mathcal{L}$ be a monotone open cover of X such that $\mathcal{L}^c = \mathcal{L}$. Then the following holds. Suppose there is a sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of open coves of X such that $\mathcal{U}_n \prec \mathcal{L}$ for each n satisfying: for each $x \in X$, there is a sequence of integers $(\langle n,x\rangle)_{n\in\mathbf{N}} \subset \mathbf{N}$ such that $\mathcal{U}^c_{\langle n+1,x\rangle}$ is a pointwise countable W-refinement of $\mathcal{U}^c_{\langle n,x\rangle}$ at x. Then $\mathcal{L}$ has a $\delta\theta$-sequence of refinements.*

Proof. This proof is similar to that of Theorem B in outline. Put $\mathcal{L} = \{W_\alpha|\alpha < \gamma\}$ for some ordinal γ. For each $V \in \bigcup_{n\in\mathbf{N}}\mathcal{U}^c_n$, define $\alpha(V) = \min\{\alpha|V \subset W_\alpha\}$. For each $V \in \bigcup_{n\in\mathbf{N}}\mathcal{U}^c_n$, define "$\mathcal{U}^c_n$ is precise at V" by the condition: If $U \in \mathcal{U}^c_n$ and $V \subset U$, then $\alpha(V) = \alpha(U)$.

For each $n \in \mathbf{N}$ and each $k \in \mathbf{N}$, put $\mathcal{W}_{n,k} = \{V \in \mathcal{U}^c_k|\mathcal{U}^c_n$ is precise at $V\}$ and $L_{n,k} = \{x \in X|\mathcal{U}^c_k$ is a pointwise countable W-refinement of $\mathcal{U}^c_n$ at $x\}$.

For each $h > 2$ and each $s = (s(1), s(2), ..., s(h)) \in \mathbf{N}^h$, define $L_s = L_{s(h-2),s(h-1)}$.

For each $x \in X$, there is a sequence $(\langle n,x\rangle)_{n\in\mathbf{N}}$ of integers such that there is a countable subfamily $\mathcal{Q}_n(x)$ of $(\mathcal{U}^c_{\langle n,x\rangle})_x$ such that $(\mathcal{U}^c_{\langle n+1,x\rangle})_x \prec \mathcal{Q}_n(x)$ for each n.

Put $Q(n,x) = \bigcup \mathcal{Q}_n(x)$.

For each $h > 2$ and each $s = (s(1), s(2), ..., s(h)) \in \mathbf{N}^h$, put $H_s = \{x \in L_s|s(i) = \langle i,x\rangle$ for $i = 1, 2, ..., h; Q(h-1,x) \in \mathcal{W}_{s(h-2),s(h-1)}\}$. Then we have

(1) $\{H_s|s \in \mathbf{N}^h, h > 2\}$ is a cover of X.

Proof. Let $x \in X$. Put $\alpha_n = \alpha(Q(n,x))$. Since $\mathcal{Q}_{n+1}(x) \prec \mathcal{Q}_n(x), Q(n+1,x) \subset Q(n,x)$. Therefore $\alpha_{n+1} \leq \alpha_n$ for each n. Thus there is a k such that $\alpha_k = \alpha_n(\forall n \geq k-2)$. Put $s = (\langle 1,x\rangle, \langle 2,x\rangle, ..., \langle k+1,x\rangle) \in \mathbf{N}^{k+1}$. Then we have

(*) $x \in H_s$.

Proof. It is obvious that $x \in L_s$ and $Q(k,x) \in \mathcal{U}^c_{\langle k,x\rangle} = \mathcal{U}^c_{s(k)}$. If $Q(k,x) \subset U, U \in \mathcal{U}^c_{\langle k-1,x\rangle}$, then $x \in U$. Thus $U \in (\mathcal{U}^c_{\langle k-1,x\rangle})_x$. Since $(\mathcal{U}^c_{\langle k-1,x\rangle})_x \prec \mathcal{Q}_{k-2}(x)$, there exists $U' \in \mathcal{Q}_{k-2}(x)$ such that $U \subset U'$. Hence $U \subset Q(k-2,x)$. Therefore $Q(k,x) \subset U \subset Q(k-2,x)$. Thus $\alpha_k \leq \alpha(U) \leq \alpha_{k-2} = \alpha_k$. Hence $\alpha(U) = \alpha_k = \alpha(Q(k,x))$. Therefore $\mathcal{U}^c_{\langle k-1,x\rangle}$ is precise at $Q(k,x)$. Thus $Q(k,x) \in \mathcal{W}_{\langle k-1,x\rangle,\langle k,x\rangle} = \mathcal{W}_{s(k-1),s(k)}$. Hence $x \in H_s$.

For each $\alpha < \gamma$ and $n, k \in \mathbf{N}$, put $V_{\alpha,n,k} = \bigcup\{W|W \in \mathcal{W}_{n,k}, \alpha(W) = \alpha\}$ and $\mathcal{V}_{n,k} = \{V_{\alpha,n,k}|\alpha < \gamma\}$. Then $\mathcal{V}_{n,k}$ is an open family in X and

(2) $\mathcal{V}_{n,k}$ is is point countable on $L_{n,k}$.

Proof. Let $x \in L_{n,k}$. Then there exists a countable subfamily $\mathcal{Q}_n^{'}$ of $(\mathcal{U}_n^c)_x$ such that $(\mathcal{U}_k^c)_x \prec \mathcal{Q}_n^{'}$.

Put $A = \{\alpha(Q)|Q \in \mathcal{Q}_n^{'}\}$. Then $\{\alpha|x \in V_{\alpha,n,k}\} \subset A$. To show this, let $\alpha < \gamma$ and $x \in V_{\alpha,n,k}$. Then there is a $W \in \mathcal{W}_{n,k}$ such that $x \in W$ and $\alpha(W) = \alpha$. Since $W \in (\mathcal{U}_k^c)_x, W \subset Q$ for some $Q \in \mathcal{Q}_n^{'}$. Since $Q \in \mathcal{U}_n^c, W \subset Q$ and $W \in \mathcal{W}_{n,k}, \alpha(W) = \alpha(Q)$. Thus $\alpha \in A$.

For each $h > 2$ and each $s = (s(1), S(2), ..., s(h)) \in \mathbf{N}^h$, put $\mathcal{V}_s = \mathcal{V}_{s(h-2),s(h-1)}, \mathcal{U}_s = \{U \in \mathcal{U}_{s(h)}^c | U \not\subseteq \cup\mathcal{V}_s\}$ and $\mathcal{O}_s = \mathcal{U}_s \cup \mathcal{V}_s$. Then
(i) $\mathcal{O}_s$ is an open cover of X,
(ii) $\mathcal{O}_s \prec \mathcal{L}$,
(iii) for each $x \in X$, by (1), there is $h > 2$ and $s \in \mathbf{N}^h$ such that $x \in H_s$. Then $\text{ord}(x, \mathcal{O}_s) \leq \omega$.

(i) and (ii) are obvious.

Proof of (iii). Since $x \in L_{s(h-2),s(h-1)}$, by (2), $\text{ord}(x, \mathcal{V}_s) \leq \omega$. Let $U \in \mathcal{U}_s$. Then $x \notin U$. If not, $U \in (\mathcal{U}_{s(h)}^c)_x$. Since $(\mathcal{U}_{s(h)}^c)_x \prec \mathcal{Q}_{(h-1)}(x), U \subset Q(h-1, x)$. Since $x \in H_s, Q(h-1, x) \in \mathcal{W}_{s(h-2),s(h-1)}$. Therefore $Q(h-1, x) \subset V_{\alpha,s(h-2),s(h-1)}$ for some $\alpha < \gamma$. Thus $U \subset V_{\alpha,s(h-2),s(h-1)} \subset \bigcup \mathcal{V}_s$. This is a contradiction because $U \in \mathcal{U}_s$. Thus $\text{ord}(x, \mathcal{U}_s) = 0$.

By (i) ‘ (iii), $\{\mathcal{O}_s | s \in \mathbf{N}^s, h > 2\}$ is a $\delta\theta$-sequence of open refinements of $\mathcal{L}$. □

Concerning θ-sequences, the following is known.

Theorem C([12, Lemma 1.3]). Let $\mathcal{U}$ be an open cover of X. Then the following are equivalent.
(1) There is a θ-sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of refinements of $\mathcal{U}$ such that $\mathcal{U}_n$ is an interior preserving open cover of X for each n.
(2) There are a sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of interior preserving open covers of X such that $\mathcal{U}_n \prec \mathcal{U}$ for each n and a closed cover $\{F_n | n \in \mathbf{N}\}$ of X such that $\mathcal{U}_n$ is point finite at each $x \in F_n$ for each n.

Concerning $\delta\theta$-sequences, the similar result of Theorem C holds.

Theorem 4. *Let $\mathcal{U}$ be an open cover of X. Then the following are equivalent.*
(1) There is a $\delta\theta$-sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of refinements of $\mathcal{U}$ such that $\mathcal{U}_n$ is an interior preserving open cover of X for each n.
(2) There are a sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of interior preserving open covers of X such that $\mathcal{U}_n \prec \mathcal{U}$ for each n and a closed cover $\{F_n | n \in \mathbf{N}\}$ of X such that $\mathcal{U}_n$ is point countable at each $x \in F_n$ for each n.

Proof. (1) $\Rightarrow$ (2). For each n, put $F_n = \{x \in X | st(x, \mathcal{U}_n) \subset \bigcup \mathcal{U}^{'}$ for some countable subfamily $\mathcal{U}^{'}$ of $\mathcal{U}\}$. Then

(i) F_n is closed in X.

Proof. Let $x \in X \setminus F_n$. Then $\text{st}(x, \mathcal{U}_n) \not\subseteq \bigcup \mathcal{U}^{'}$ for each countable subfamily $\mathcal{U}^{'}$ of $\mathcal{U}$. Put $U_x = \bigcap (\mathcal{U}_n)_x$. Then $x \in U_x$ and, since $\mathcal{U}$ is interior preserving, U_x is open. And we have
(*) $U_x \subset X \setminus F_n$.

Proof. Let $y \in U_x$. If $U \in (\mathcal{U}_n)_x$, then $y \in U$. Therefore $(\mathcal{U}_n)_x \subset (\mathcal{U}_n)_y$. Thus $\text{st}(x, \mathcal{U}_n) \subset \text{st}(y, \mathcal{U}_n)$. Since $\text{st}(x, \mathcal{U}_n) \not\subseteq \cup\mathcal{U}^{'}$ for each countable subfamily $\mathcal{U}^{'}$ of $\mathcal{U}$. Therefore $\text{st}(y, \mathcal{U}_n) \not\subseteq \bigcup \mathcal{U}^{'}$ for each countable subfamily $\mathcal{U}^{'}$ of $\mathcal{U}$. Hence $y \notin F_n$.

Therefore $(\mathcal{U}_n)$ and F_n satisfy the conditions in (2).

(2) $\Rightarrow$ (1) is obvious. $\square$.

Theorem 5. *Let $\mathcal{U}$ be an open cover of X. If $\mathcal{U}^c$ has a $\delta\theta$-sequence of refinements, then $\mathcal{U}$ has a $\delta\theta$-sequence of refinements.*

Proof. Let $(\mathcal{V}_n)_{n\in\mathbf{N}}$ be a $\delta\theta$-sequence of refinements of $\mathcal{U}^c$. For each $V \in \mathcal{V}_n$, there exists a countable subfamily $\mathcal{U}_V$ of $\mathcal{U}^c$ such that $V \subset \bigcup\mathcal{U}_V$. Let $\mathcal{U}_V = \{U_i | i = 1, 2, ...\}, U_i = \cup_{j=1}^{\infty} U_{i,j}, U_{i,j} \in \mathcal{U}$. Put $\mathcal{U}_V^{'} = \{U_{i,j} | i, j = 1, 2, ...\}$. Define $\widetilde{\mathcal{V}_n} = \{V \cap U | U \in \mathcal{U}_V^{'}, V \in \mathcal{V}_n\}$. Then $\widetilde{\mathcal{V}_n}$ is an open cover of X and $\overline{\mathcal{V}_n} \prec \mathcal{U}$. For each $x \in X$, there is an n such that $1 \le \mathrm{ord}(x, \mathcal{V}_n) \le \omega$. Then $1 \le \mathrm{ord}(x, \widetilde{\mathcal{V}_n}) \le \omega$. Thus $(\widetilde{\mathcal{V}_n})_{n\in\mathbf{N}}$ is a $\delta\theta$-sequence of refinements of $\mathcal{U}$. $\square$

Definition 4. Let $\mathcal{U}$ be a cover of X and $(\mathcal{V}_n)_{n\in\mathbf{N}}$ a sequence of covers of X. A sequence $(\mathcal{V}_n)_n$ is called a pointwise W-refining sequence for $\mathcal{U}$ if for each x, there exists some n_x such that $\mathcal{V}_{n_x}$ is a pointwise W-refinement of $\mathcal{U}$ at x.

By Worrell, the next characterization of θ-refinable spaces was given.

Theorem D ([18], or cf. [19, 3.4. Theorem]). A space X is θ-refinable (submetacompact) if and only if every open cover of X has a pointwise W-refining sequence by open covers.

Definition 5. ([11]). Let $\mathcal{L}$ and $\mathcal{G}$ be covers of X. $\mathcal{L}$ is called "*point-star $\dot{F}$-refinement*" of $\mathcal{G}$ at $x \in X$ if there is a finite subfamily $\mathcal{G}^{'}$ of $\mathcal{G}$ such that $x \in \bigcap\mathcal{G}^{'}$ and $\mathrm{st}(x, \mathcal{L}) \subset \bigcup\mathcal{G}^{'}$.

A sequence $(\mathcal{L}_n)_{n\in\mathbf{N}}$ of covers of X is called "*point-star $\dot{F}$-refining sequence*" of $\mathcal{G}$ if for each $x \in X$, there is an $n_x \in \mathbf{N}$ such that $\mathcal{L}_{n_x}$ is point-star $\dot{F}$-refinement of $\mathcal{G}$ at x.

Junnila gave the next characterization of submetacompactness.

Theorem E ([18]). A space X is θ-refinable (submetacompact) if and only if every open cover of X has a point star $\dot{F}$-refinning sequence by open covers.

Definition 6. Let $\mathcal{U}$ be a cover of X and $(\mathcal{V}_n)_{n\in\mathbf{N}}$ a sequence of covers of X. We shall say a sequence $(\mathcal{V}_n)_n$ is a pointwise countable W-refining sequence for $\mathcal{U}$ if for each x, there exists some n_x such that $\mathcal{V}_{n_x}$ is a pointwise countable W-refinement of $\mathcal{U}$ at x.

We shall say a space X is w-$\delta\theta$-refinable if every open cover of X has a pointwise countable W-refining sequences by open covers.

Definition 7. Let $\mathcal{L}$ and $\mathcal{G}$ are covers of X. We shall say $\mathcal{L}$ is called "*point-star $\dot{C}$-refinement*" of $\mathcal{G}$ at $x \in X$ if if there is a countable subfamily $\mathcal{G}^{'}$ of $\mathcal{G}$ such that $x \in \bigcap\mathcal{G}^{'}$ and $\mathrm{st}(x, \mathcal{L}) \subset \bigcup\mathcal{G}^{'}$.

We shall say a sequence $(\mathcal{L}_n)_{n\in\mathbf{N}}$ of covers of X is "*point-star $\dot{C}$-refining sequence*" of $\mathcal{G}$ if for each $x \in X$, there is an $n_x \in \mathbf{N}$ such that $\mathcal{L}_{n_x}$ is point-star $\dot{C}$-refinement of $\mathcal{G}$ at x.

We shall say a space X is ww-$\delta\theta$-refinable if every open cover of X has a point star $\dot{C}$-refining sequences by open covers.

It is obvious that every $\delta\theta$-refinable space is w-$\delta\theta$-refinable and every w-$\delta\theta$-refinable space is ww-$\delta\theta$-refinable. Let $L(X)$ denote the Lindelöf number of a space X, i.e., $L(X) = \min\{\kappa \mid \kappa \ge \omega$, each open cover $\mathcal{G}$ of X has a subcover $\mathcal{G}^{'}$ with $|\mathcal{G}^{'}| \le \kappa\}$.

Theorem 6. *Let X be a space with $L(X) \le \omega_1$. Then the following are equivalent.*
(i) X is $\delta\theta$-refinable.
(ii) X is w-$\delta\theta$-refinable.
(iii) X is ww-$\delta\theta$-refinable.

Proof. It is obvious that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). To prove that (iii) $\Rightarrow$ (i), let $\mathcal{U}$ be an open cover of X. We may assume that $\mathcal{U} = \{U_\alpha | \alpha < \omega_1\}$. By assumption, there exists a sequence $(\mathcal{L}_k)_{k\in\mathbf{N}}$ of point star $\dot{C}$-refining sequence by open covers of X.

For each $k \in \mathbf{N}$ and each $\alpha < \omega_1$, define
$V_{k,\alpha} = U_\alpha \cap (\mathrm{st}(X \setminus \bigcup_{\beta\neq\alpha} U_\beta, \mathcal{L}_k))$,
$V'_{k,\alpha} = U_\alpha \cap (\bigcup_{\beta>\alpha} U_\beta) \cap (\mathrm{st}(X \setminus \bigcup_{\beta<\alpha} U_\beta, \mathcal{L}_k))$ and put
$\mathcal{V}_k = \{V_{k,\alpha} | \alpha < \omega_1\} \cup \{V'_{k,\alpha} | \alpha < \omega_1\}$.
Then

(1) $\mathcal{V}_k$ is an open cover of X such that $\mathcal{V}_k \prec \mathcal{U}$.

Proof. It is obvious that each set of $\mathcal{V}_k$ is an open set and $\mathcal{V}_k \prec \mathcal{U}$. To prove that $\mathcal{V}_k$ is a cover of X, let $x \in X$. Put $\alpha = \min\ \{\beta < \omega_1 | x \in U_\beta\}$. Then $x \in U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. If $x \notin V'_{k,\alpha}$, then $x \notin \bigcup_{\beta>\alpha} U_\beta$ and thus $x \in \bigcup_{\beta\neq\alpha} U_\beta$. Hence $x \in V_{k,\alpha}$.
(2) $(\mathcal{V}_k)_{k\in\mathbf{N}}$ is a $\delta\theta$-sequence.
Proof. Let $x \in X$. Then there exist a $k \in \mathbf{N}$ and a countable subset $\{\alpha_i | i = 1, 2, ...\} \subset \omega_1$ such that $x \in \bigcap_{i=1}^\infty U_{\alpha_i}$ and $\mathrm{st}(x, \mathcal{L}_k) \subset \bigcup_{i=1}^\infty U_{\alpha_i}$.

If $x \in V_{k,\alpha}$, then there is an $L \in \mathcal{L}_k$ such that $x \in L$ and $L \cap (X \setminus \bigcup_{\beta\neq\alpha} U_\beta) \neq \emptyset$. Since $L \subset \bigcup_{i=1}^\infty U_{\alpha_i}, \alpha = \alpha_i$ for some i. Therefore $\{\alpha < \omega_1 | x \in V_{k,\alpha}\} \subset \{\alpha_i | i = 1, 2, ...\}$. Put $\alpha^* = \sup\{\alpha_i | i = 1, 2, ...\}$. Then $\{\alpha < \omega_1 | x \in V'_{k,\alpha}\} \subset \{\alpha | \alpha \leq \alpha^*\}$. To show this, let $\alpha > \alpha^*$. If $x \in L \in \mathcal{L}_k$, then $L \subset \bigcup_{\beta<\alpha} U_\beta$. Thus $x \notin V'_{k,\alpha}$. Hence $\mathrm{ord}(x, \mathcal{V}_k) \leq \omega$. $\square$

3. $\delta\theta$-refinability-like properties of σ-products

Throughout this sectuion we assume that each space is a T_1-space having at least two points. We define σ-products which were introduced by H. H. Corson [8].

Definition 8. Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces. "$\sigma = \sigma(\mathcal{S})$ is a σ-product of $\mathcal{S}$" means there is a point $x^* = (x^*_\alpha)_{\alpha\in\Omega} \in X = \Pi\{X_\alpha | \alpha \in \Omega\}$ (called the base point of σ) such that σ is the subspace of X consisting of $\{x \in X | Q(x) \text{ is finite}\}$. Here $Q(x) = \{\alpha | \alpha \in \Omega, x_\alpha \neq x^*_\alpha\}$. Let $\Omega^n = \{a \subset \Omega : |a| = n\}$ each $n \in \omega$ and put $\Omega^{<\omega} = \cup\{\Omega^n | n \in \omega\}$. Here $|a|$ denotes the cardinal number of a.

For a finite subset F of $\Omega, \Pi\{X_\alpha | \alpha \in F\}$ is said to be a finite subproduct of σ.

For each $a \in \Omega^{<\omega}$, define $Y_a = \Pi_{\alpha\in a} X_\alpha \times \{x^*_\alpha\}_{\alpha\in\Omega\setminus a}$. Let $p_a : \sigma \to Y_a$ be the map defined by

$$p_a(x)_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in a \\ x^*_\alpha & \text{if } \alpha \in \Omega \setminus a. \end{cases}$$

Then p_a is an open continuous onto map.

For each $x \in \sigma$, put $x_a = p_a(x)$.

The following fact concerning σ-products is known.

Fact. Let $\sigma = \sigma(\mathcal{S})$ and $\sigma_n = \{x \in \sigma : |Q(x)| \leq n\}$ for each $n \in \omega$. Then σ_n is closed in σ.

Several papers have investigated the results for σ-products of the following type:
(*) Let $\mathcal{P}$ be a topological property. Let σ be a σ-product of spaces. If each finite subproduct of σ has property $\mathcal{P}$, then σ has $\mathcal{P}$.

First, Kombarov [15] proved that (*) holds for $\mathcal{P}$ being paracompactness and Lindelöfness for regular spaces. After that, it was proved that (*) holds for $\mathcal{P}$ being the following properties: Lindelöfness (Chiba [6]), metacompactness (Teng [17]), subparacompactness and θ-refinability (submetacompactness)([17]) , weak θ-refinability, weak $\delta\theta$-refinability, hereditarily weak θ-refinability and hereditarily weak $\delta\theta$-refinability ([5]).

Concerning $\delta\theta$-refinability (submeta-Lindelöfness), the following is known.

Theorem F ([5]). Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is normal. If every finite subproduct of σ is $\delta\theta$-refinable, then σ is $\delta\theta$-refinable.

In this paper we investigate $\delta\theta$-refinability and $\delta\theta$-refinability-like properties of σ-products.

Let κ be an infinite cardinal. A space X is called κ-paracompact if every open cover of X with its cardinality $\leq \kappa$ has a locally finite open refinement.

A space X is called κ-subparacompact if every open cover of X with its cardinality $\leq \kappa$ has a σ-locally finite closed refinement.

A space X is called κ-submetacompact if every open cover of X with its cardinality $\leq \kappa$ has a θ-sequence of open refinements.

A space X is *subnormal* if for any disjoint closed sets A and B in X, there are disjoint G_δ-sets G and H such that $A \subset G$ and $B \subset H$.

Lemma 1. *([2]). A space X is κ-subparacompact if and only if for every cover of X with its cardinality $\leq \kappa$ has a σ-discrete closed refinement.*

Lemma 2. *([7, Lemma 2.5]). A space X is subnormal and κ-paracompact, then X is κ-subparacompact.*

Theorem 7. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is subnormal and κ-paracompact where $\kappa = |\Omega|$. If every finite subproduct of σ is $\delta\theta$-refinable, then σ is $\delta\theta$-refinable.*

By Lemma 2, Theorem 7 follows from Theorem 8 below.

Theorem 8. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-paracompact and κ-subparacompact where $\kappa = |\Omega|$. If every finite subproduct of σ is $\delta\theta$-refinable, then σ is $\delta\theta$-refinable.*

Proof. Let $\mathcal{A} = \Omega^{<\omega}$ and put $\Lambda = \mathcal{A}^{<\omega}$. Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an arbitrary open cover of σ. For each $a \in \mathcal{A}$, let $U_{a,\xi}$ be the maximal open set in Y_a satisfying $p_a^{-1}(U_{a,\xi}) \subset G_\xi$ and put $U_a = \cup_{\xi \in \Xi} U_{a,\xi}$. Then $\{p_a^{-1}(U_a) | a \in \mathcal{A}\}$ is an open cover of σ such that $p_a^{-1}(U_a) \subset p_b^{-1}(U_b)$ for each $a, b \in \mathcal{A}$ with $a \subset b$. Since $|\mathcal{A}| = \kappa$ and σ is κ-paracompact, there is a locally finite open cover $\mathcal{J} = \{J_a | a \in \mathcal{A}\}$ of σ such that $J_a \subset p_a^{-1}(U_a)$ for each $a \in \mathcal{A}$. For each $\lambda \in \Lambda$, let us put $V_\lambda = \sigma \setminus \cup_{b \in \mathcal{A} \setminus \lambda} \overline{J_b}$. The we have:
(1) $\mathcal{V} = \{V_\lambda | \lambda \in \Lambda\}$ is an open cover of σ.
(2) $V_\lambda \subset V_\nu$ if $\lambda, \nu \in \Lambda$ with $\lambda \subset \nu$.
(3) Put $a_\lambda = \cup\{a | a \in \lambda\}$. Then $a_\lambda \in \mathcal{A}$ and $\overline{V_\lambda} \subset p_{a_\lambda}^{-1}(U_{a_\lambda})$.

For each $\lambda \in \Lambda$, define $T_{a_\lambda} = Y_{a_\lambda} \setminus p_{a_\lambda}(\sigma \setminus \overline{V_\lambda})$ and put $C_\lambda = \mathrm{Int} p_{a_\lambda}^{-1}(T_{a_\lambda})$. Then T_{a_λ} is a closed subset of Y_{a_λ} and we have
(4) $T_{a_\lambda} \subset U_{a_\lambda}$ for each $\lambda \in \Lambda$.
(5) $\mathcal{C} = \{C_\lambda | \lambda \in \Lambda\}$ is an open cover of σ. (This was essentially proved in [1], or see [4]).

Since σ is κ-subparacompact and $|\Lambda| = \kappa$, there is a σ-discrete closed cover $\mathcal{F} = \bigcup_{n \in \mathbf{N}} \mathcal{F}_n$ of σ, where $\mathcal{F}_n$ is discrete in σ such that $\mathcal{F}_n \prec \mathcal{C}$. We can represent $\mathcal{F}_n = \{F_{\lambda,n} | \lambda \in \Lambda\}$ with $F_{\lambda,n} \subset C_\lambda$ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda, \mathcal{U}_\lambda = \{U_{a_\lambda,\xi} | \xi \in \Xi\}$ is an open cover of U_{a_λ}. Since Y_{a_λ} is $\delta\theta$-refinable and T_{a_λ} is closed in Y_{a_λ}, there is a sequence $(\mathcal{H}_{\lambda,m})_{m \in \mathbf{N}}$ of collections of open sets in Y_{a_λ} satisfying:
$(6)_\lambda$. $\mathcal{H}_{\lambda,m} \prec \mathcal{U}_\lambda$ for each m.

$(7)_\lambda$. $\mathcal{H}_{\lambda,m}$ covers T_{a_λ} for each m.
$(8)_\lambda$. For each $y \in T_{a_\lambda}$, there is an $m(y) \in \mathbf{N}$ such that $\mathrm{ord}(y, \mathcal{H}_{\lambda,m(y)}) \le \omega$.

Here we can represent $\mathcal{H}_{\lambda,m} = \{H_{\lambda,m,\xi} | \xi \in \Xi\}$ with $H_{\lambda,m,\xi} \subset U_{a_\lambda,\xi}$ for each $\xi \in \Xi$.

For each $n \in \omega, n \in \mathbf{N}, \lambda \in \Lambda$ and $\xi \in \Xi$, let $H(n,m,\lambda,\xi) = p_{a_\lambda}^{-1}(H_{\lambda,m,\xi}) \cap C_\lambda \cap (\sigma \setminus \cup_{\mu \ne \lambda} F_{\mu,n})$ and put $\mathcal{H}_{n,m} = \{H(n,m,\lambda,\xi) | \lambda \in \Lambda, \xi \in \Xi\}$. Then we have:
(9) $\mathcal{H}_{n,m}$ is an open cover of σ.
(10) $\mathcal{H}_{n,m} \prec \mathcal{G}$.
(11) For each $x \in \sigma$, there are an $n \in \omega$ and an $m \in \mathbf{N}$ such that $\mathrm{ord}(x, \mathcal{H}_{n,m}) \le \omega$.

Proof of (9). Let $x \in \sigma$. If $x \notin \bigcup \mathcal{F}_n$, then $x \in \sigma \setminus \bigcup \mathcal{F}_n$. By (5), $x \in C_\lambda$ for some λ. Then $x_{a_\lambda} \in T_{a_\lambda}$. By $(7)_\lambda$, $x_{a_\lambda} \in H_{\lambda,m,\xi}$ for some ξ. Thus $x \in H(n,m,\lambda,\xi)$.

If $x \in \bigcup \mathcal{F}_n$, then $x \in F_{\lambda,n}$ for some $\lambda \in \Lambda$. Since $\mathcal{F}_n$ is discrete, $x \notin \cup_{\mu \ne \lambda} F_{\mu,n}$. Since $F_{\lambda,n} \subset C_\lambda, x \in C_\lambda$. Therefore $x \in H(n,m,\lambda,\xi)$ for some ξ.

Proof of (10). Let $H_{\lambda,m,\xi} \in \mathcal{H}_{\lambda,m}$. Then $H_{\lambda,m,\xi} \subset U_{a_\lambda,\xi}$. Thus $p_{a_\lambda}^{-1}(H_{\lambda,m,\xi}) \subset G_\xi$. Hence $H(n,m,\lambda,\xi) \subset G_\xi$.

Proof of (11). Let $x \in \sigma$. Since $\mathcal{F}$ is a cover of σ, there are an $n \in \omega$ and a $\lambda \in \Lambda$ such that $x \in F_{n,\lambda}$. Then $x \notin \cup_{\mu \ne \lambda} F_{\mu,n}$ and $x \in C_\lambda$. Thus $x_{a_\lambda} \in T_{a_\lambda}$. By $(8)_\lambda$, there is an m such that $\mathrm{ord}(x_{a_\lambda}, \mathcal{H}_{\lambda,m}) \le \omega$. Then $\mathrm{ord}(x, \mathcal{H}_{n,m}) \le \omega$. To show this, let $x \in H(n,m,\lambda,\xi)$. Then $x_{a_\lambda} \in H_{\lambda,m,\xi}$. Such λ are at most countable.

Thus $\{\mathcal{H}_{n,m} | n \in \omega, m \in \mathbf{N}\}$ is a $\delta\theta$-sequence of open refinements of $\mathcal{G}$.

Remark 1 ([7, p.85, Remark]). As is well-known, paracompactness implies subparacompactness. However, for each $\lambda \ge \omega$, λ-paracompactness does not imply λ-paracompactness.

The author proved in [4] that under the assumption of σ being κ-paracompact, if every finite subproduct of σ is normal, then σ is normal. We can prove the following similarly.

Theorem 9. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-paracompact where $\kappa = |\Omega|$. If every finite subproduct of σ is subnormal, then σ is subnormal.*

Proof. Let $\mathcal{G} = \{G_i | i = 1,2\}$ be an arbitrary binary open cover of σ. Let us define $\mathcal{A}, \Lambda, U_{a,i}, U_a, \mathcal{J}, V_\lambda, T_{a_\lambda}, C_\lambda$ and $\mathcal{U}_\lambda$ are similar to that of the proof of Theorem 8.

For each $\lambda \in \Lambda, \mathcal{U}_\lambda = \{U_{a_\lambda,i} | i = 1,2\}$ is an open cover of U_{a_λ}. Since Y_{a_λ} is subnormal, there are F_σ-sets $K_{\lambda,i}, i = 1,2$ of T_{a_λ} such that $T_{a_\lambda} = \cup_{i=1}^2 K_{\lambda,i}$ and $K_{\lambda,i} \subset U_{a_\lambda,i}$ for $i = 1,2$. Let $\mathcal{O} = \{O_\lambda | \lambda \in \Lambda\}$ be a locally finite open cover of σ such that $O_\lambda \subset C_\lambda$ for each $\lambda \in \Lambda$.

Let us put $K_i = \bigcup_{\lambda \in \Lambda} (p_{a_\lambda}^{-1}(K_{\lambda,i}) \cap \overline{O_\lambda})$. Then K_i are F_σ-sets in $\sigma, K_i \subset G_i$ for $i = 1,2$ and $\sigma = \cup_{i=1}^2 K_i$. □

By Theorems 7 and 9, we obtain the following.

Theorem 10. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-paracompact where $\kappa = |\Omega|$. If every finite subproduct of σ is subnormal and $\delta\theta$-refinable, then σ is $\delta\theta$-refinable.*

Lemma 3. *(1) Let $\mathcal{G}$ be an open cover of X and $(\mathcal{V}_n)_{n \in \mathbf{N}}$ is a pointwise countable W-refining sequence of $\mathcal{G}$. Then there exists a pointwise countable W-refining sequence $(\mathcal{H}_n)_{n \in \mathbf{N}}$ of $\mathcal{G}$ satisfying the following conditions: For each $x \in X$, there exist an $n_x \in \mathbf{N}$ and a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $(\mathcal{H}_n)_x \prec \mathcal{G}'$ for each $n \ge n_x$.*
(2) Let $\mathcal{G}$ be an open cover of X and $(\mathcal{V}_n)_{n \in \mathbf{N}}$ is a point-star $\dot{C}$-refining sequence of $\mathcal{G}$. Then there exists a point-star $\dot{C}$-refining sequence $(\mathcal{H}_n)_{n \in \mathbf{N}}$ of $\mathcal{G}$ satisfying the following

conditions: For each $x \in X$, there exist an $n_x \in \mathbf{N}$ and a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $x \in \bigcap \mathcal{G}'$ and $st(x, \mathcal{H}_n) \subset \bigcup \mathcal{G}'$ for each $n \geq n_x$.

Proof. Let us put $\mathcal{H}_n = \wedge_{i=1}^n \mathcal{V}_i (= \{\cap_{i=1}^n V_i | V_i \in \mathcal{V}_i$ for each $i = 1, 2, ..., n\})$. Then $(\mathcal{H}_n)_{n\in\mathbf{N}}$ is a desired one. □

Theorem 11. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-paracompact where $\kappa = |\Omega|$. If every finite subproduct of σ is w-$\delta\theta$-refinable, then σ is w-$\delta\theta$-refinable.*

Proof. Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an arbitrary open cover of σ. Let us define $\mathcal{A}, \Lambda, U_{a,\xi}, U_a, \mathcal{J}, V_\lambda, T_{a_\lambda}, C_\lambda$ and $\mathcal{U}_\lambda$ are similar to that of the proof of Theorem 8.

Since $|\Lambda| = \kappa$ and σ is κ-paracompact, there is a locally finite open cover $\mathcal{O} = \{O_\lambda | \lambda \in \Lambda\}$ of σ such that $O_\lambda \subset C_\lambda$ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda, \mathcal{U}_\lambda = \{U_{a_\lambda,\xi} | \xi \in \Xi\}$ is an open cover of U_{a_λ}. Since Y_{a_λ} is w-$\delta\theta$-refinable and T_{a_λ} is closed in Y_{a_λ}, there is a sequence $(\mathcal{H}_{\lambda,m})_{m\in\mathbf{N}}$ of collections of open sets in Y_{a_λ} satisfying:
$(6)_\lambda$. $\mathcal{H}_{\lambda,m}$ covers T_{a_λ} for each m.
$(7)_\lambda$. For each $y \in T_{a_\lambda}$, there are a countable subset Ξ_y of Ξ and an $m_y \in \mathbf{N}$ such that $\mathcal{H}_{\lambda,m}(y)$ is a partial refinement of $\{U_{a_\lambda,\xi} | \xi \in \Xi_y\}$ for each $m \geq m_y$.

Put $\mathcal{H}_m = \{p_{a_\lambda}^{-1}(H) \cap O_\lambda | H \in \mathcal{H}_{\lambda,m}, \lambda \in \Lambda\}$. Then we have:
(8) $\mathcal{H}_m$ is an open cover of σ.
(9) For each $x \in \sigma$, there are a countable subset Ξ_x of Ξ and an $m_x \in \mathbf{N}$ such that $\mathcal{H}_{m_x}(x)$ is a partial refinement of $\{G_\xi | \xi \in \Xi_x\}$.

Proof of (8). Let $x \in \sigma$. Then $x \in O_\lambda$ for some λ. Therefore $x_{a_\lambda} \in T_{a_\lambda}$. By $(6)_\lambda, x_{a_\lambda} \in H$ for some $H \in \mathcal{H}_{\lambda,m}$. Thus $x \in p_{a_\lambda}^{-1}(H) \cap O_\lambda$.

Proof of (9). Let $x \in \sigma$. Since $\mathcal{O}$ is locally finite, there is a finite subset $\{\lambda_i | i = 1, 2, ..., n\}$ such that $x \in O_\lambda \iff \lambda \in \{\lambda_i | i = 1, 2, ..., n\}$. For each $i = 1, 2, ..., n$, since $x_{a_{\lambda_i}} \in T_{a_{\lambda_i}}$, there are countable subsets Ξ_i of Ξ and $m_i \in \mathbf{N}$ for $i = 1, 2, ..., n$ such that $\mathcal{H}_{\lambda_i,m}(x_{a_{\lambda_i}})$ is a partial refinement of $\{U_{a_{\lambda_i},\xi} | \xi \in \Xi_i\}$ for every $m \geq m_i$. Let us put $m^* = \max\{m_i | i = 1, 2, ..., n\}$ and $\Xi^* = \cup_{i=1}^n \Xi_i$. Then $\mathcal{H}_m(x)$ is a partial refinement of $\{G_\xi | \xi \in \Xi^*\}$.

To show this, let $x \in p_{a_\lambda}^{-1}(H) \cap O_\lambda, H \in \mathcal{H}_{\lambda,m}$. Then $\lambda = \lambda_i$ for some $i = 1, 2, ..., n$. Since $x_{a_{\lambda_i}} \in H, H \subset U_{a_{\lambda_i},\xi}$ for some ξ_i. Therefore $p_{a_{\lambda_i}}^{-1}(U_{a_{\lambda_i},\xi}) \subset G_\xi$. □

Theorem 12. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-paracompact where $\kappa = |\Omega|$. If every finite subproduct of σ is ww-$\delta\theta$-refinable, then σ is ww-$\delta\theta$-refinable.*

Proof. Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an arbitrary open cover of σ. Let us define $\Lambda, U_{a,\xi}, U_a, V_\lambda, T_{a_\lambda}, C_\lambda, O_\lambda$ and $\mathcal{U}_\lambda$ are similar to that of the proof of Theorem 11. Since Y_{a_λ} is ww-$\delta\theta$-refinable and T_{a_λ} is closed in Y_{a_λ}, there is a sequence $(\mathcal{H}_{\lambda,m})_{m\in\mathbf{N}}$ of collections of open sets in Y_{a_λ} satisfying:
$(6)_\lambda$. $\mathcal{H}_{\lambda,m}$ covers T_{a_λ} for each m.
$(7)'_\lambda$. For each $y \in T_{a_\lambda}$, there are a countable subset Ξ_y of Ξ and an $m_y \in \mathbf{N}$ such that
(i). $y \in \bigcap\{U_{a_\lambda,\xi} | \xi \in \Xi_y\}$,
(ii). $\mathrm{st}(y, \mathcal{H}_{\lambda,m}) \subset \bigcup\{U_{a_\lambda,\xi} | \xi \in \Xi_y\}$ for each $m \geq m_y$.

Put $\mathcal{H}_m = \{p_{a_\lambda}^{-1}(H) \cap O_\lambda | H \in \mathcal{H}_{\lambda,m}, \lambda \in \Lambda\}$. Then we have:
(8) $\mathcal{H}_m$ is an open cover of σ.
$(9)'$ For each $x \in \sigma$, there are a countable subset Ξ_x of Ξ and an $m_x \in \mathbf{N}$ such that
(i) $x \in \bigcap\{G_\xi | \xi \in \Xi_x\}$,
(ii) $\mathrm{st}(x, \mathcal{H}_{m_x}) \subset \bigcup\{G_\xi | \xi \in \Xi_x\}$. □

Theorem 13. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and G an open subspace of $\sigma = \sigma(\mathcal{S})$. Suppose G is κ-submetacompact where $\kappa = |\Omega|$. If every finite subproduct of σ is hereditarily w-$\delta\theta$-refinable, then G is w-$\delta\theta$-refinable.*

Proof. Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an arbitrary open cover of G. For each $a \in \mathcal{A}$, let $U_{a,\xi}$ be the maximal open set in Y_a satisfying $p_a^{-1}(U_{a,\xi}) \subset G_\xi$ and put $U_a = \bigcup_{\xi \in \Xi} U_{a,\xi}$. Since $\mathcal{U} = \{p_a^{-1}(U_a) | a \in \mathcal{A}\}$ is an open cover of G with $|\mathcal{U}| = \kappa$, there is a σ-discrete closed cover $\mathcal{F} = \bigcup_{n \in \mathbf{N}} \mathcal{F}_n$ of G, where $\mathcal{F}_n$ is discrete in G such that $\mathcal{F}_n \prec \mathcal{U}$. We can represent $\mathcal{F}_n = \{F_{a,n} | a \in \mathcal{A}\}$ with $F_{a,n} \subset U_a$ for each $a \in \mathcal{A}$.

For each $a \in \mathcal{A}$, since $\mathcal{U}_a = \{U_{a,\xi} | \xi \in \Xi\}$ is an open cover of U_a and U_a is $\delta\theta$-refinable, there is a sequence $(\mathcal{H}_{a,m})_{m \in \mathbf{N}}$ of open covers of U_a satisfying:
$(1)_a$. $\mathcal{H}_{a,m} \prec \mathcal{U}_a$ for each m.
$(2)_a$. For each $y \in U_a$, there is an $m(y) \in \mathbf{N}$ such that $\mathrm{ord}(y, \mathcal{H}_{a,m(y)}) \leq \omega$.

Here we can represent $\mathcal{H}_{a,m} = \{H_{a,m,\xi} | \xi \in \Xi\}$ with $H_{a,m,\xi} \subset U_{a,\xi}$ for each $\xi \in \Xi$.

For each $n \in \omega, m \in \mathbf{N}, a \in \mathcal{A}$ and $\xi \in \Xi$, let $H(n,m,a,\xi) = p_a^{-1}(H_{a,m,\xi}) \cap (G \setminus \bigcup_{b \in \mathcal{A}, b \neq a} F_{b,n})$ and put $\mathcal{H}_{n,m} = \{H(n,m,a,\xi) | a \in \mathcal{A}, \xi \in \Xi\}$. Then we have:
(3) $\mathcal{H}_{n,m}$ is an open cover of G.
(4) $\mathcal{H}_{n,m} \prec \mathcal{G}$.
(5) For each $x \in G$, there are an $n \in \omega$ and an $m \in \mathbf{N}$ such that $\mathrm{ord}(x, \mathcal{H}_{n,m}) \leq \omega$.
Thus $\{\mathcal{H}_{n,m} | n \in \omega, m \in \mathbf{N}\}$ is a $\delta\theta$-sequence of refinements of $\mathcal{G}$. □

Theorem 14. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and G an open subspace of $\sigma = \sigma(\mathcal{S})$. Suppose G is κ-submetacompact where $\kappa = |\Omega|$. If every finite subproduct of σ is hereditarily ww-$\delta\theta$-refinable, then G is ww-$\delta\theta$-refinable.*

Proof. This proof is similar to that of Theorem 13. □

Corollary 1. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-submetacompact where $\kappa = |\Omega|$. If every finite subproduct of σ is hereditarily w-$\delta\theta$-refinable, then σ is w-$\delta\theta$-refinable.*

Corollary 2. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be spaces and $\sigma = \sigma(\mathcal{S})$. Suppose σ is κ-submetacompact where $\kappa = |\Omega|$. If every finite subproduct of σ is hereditarily ww-$\delta\theta$-refinable, then σ is ww-$\delta\theta$-refinable.*

4. Appendix to σ-products

Let us consider the following conditions for a space X.
(S_1) X has an increasing closed cover $\{X_n | n \in \omega\}$.
(S_2) For each $n \in \omega$, there is a closed cover $\mathcal{Y}_n = \{Y_a | a \in A_n\}$ of X_n.
(S_3) For each $a \in A = \bigcup_{n \in \omega} A_n$, there is a continuous onto map $p_a : X \to Y_a$ such that $p_a | Y_a = \text{identity}$.
(S_4) For each $n \in \omega$ and each open set U such that $X_{n-1} \subset U$, there is a discrete family $\mathcal{J} = \{J_a | a \in A_n\}$ of open sets in X such that $J_a \supset Y_a \setminus U$. Here $X_{-1} = \emptyset$.
(S_5) $\mathcal{K}_n = \{Y_a \setminus X_{n-1} | a \in A_n\}$ is a discrete family of closed subsets in $X \setminus X_{n-1}$ for each $n \in \omega$. Here $X_{-1} = \emptyset$.
(S_6) There is a point finite open expansion of $\mathcal{K}_n$ in X for each $n \in \omega$ (i.e., there is a point finite open family $\mathcal{M}_n = \{M_{n,a} | a \in A_n\}$ in X such that $M_{n,a} \supset Y_a \setminus X_{n-1}$ for each $a \in A_n$.

Each normal σ-product space satisfies the conditions $(S_1) \sim (S_6)$. Each σ-product space and each open subspace of it satisfies the conditions $(S_1) \sim (S_3)$ and $(S_5) \sim (S_6)$.

In [5], the author generalised the theorems of the type: "(*) Let $\mathcal{P}$ be a topological property. Let σ be a σ-product of spaces. If each finite subproduct of σ has property $\mathcal{P}$, then σ has $\mathcal{P}$." to the theorem of the type:
(1) Suppose X satisfies the conditions $(S_1) \sim (S_4)$. If each Y_a has the property P, then X has the property P.
(2) Suppose X satisfies the conditions $(S_1) \sim (S_3)$ and $(S_5) \sim (S_6)$. If each Y_a has the property P, then X has the property P.
The results of metacompactness and submetacompactness of σ-products are generalized to the following by the same proof of [16] and [17],

Theorem 15. *([16]). Suppose X satisfies the conditions $(S_1) \sim (S_3)$ and $(S_5) \sim (S_6)$. If each Y_a is metacompact, then X is metacompact.*

Theorem 16. *([17]). Suppose X satisfies the conditions $(S_1) \sim (S_3)$ and $(S_5) \sim (S_6)$. If each Y_a is submetacompact, then X is submetacompact.*

Remark 2. Similar result hold for metaLindelöfness.

Definition 9. A space X is called "discretely θ-expandable" [14] if for every discrete collection $\{F_\xi | \xi \in \Xi\}$ of subsets of X, there exists a sequence $(\mathcal{G}_n = \{G_{\xi,n} | \xi \in \Xi\})_{n \in \mathbf{N}}$ of collections of open subsets of X satisfying the following:
(i) $F_\xi \subset G_{\xi,n}$ for each ξ and each n.
(ii) For every point x of X there is n_x for which x is contained in at most finite member of $\mathcal{G}_{n_x}$ (i.e., $\mathcal{G}_{n_x}$ is point finite at x).

A space X is called "θ-expandable" [14] if for every locally finite collection $\{F_\xi | \xi \in \Xi\}$ of subsets of X, there exists a sequence $(\mathcal{G}_n = \{G_{\xi,n} | \xi \in \Xi\})_{n \in \mathbf{N}}$ of collections of open subsets of X satisfying the following:
(i) $F_\xi \subset G_{\xi,n}$ for each ξ and each n.
(ii) For every point x of X there is an n_x for which x is contained in at most finite member of $\mathcal{G}_{n_x}$ (i.e., $\mathcal{G}_{n_x}$ is point finite at x).

Theorem G ([5, Proposition 2]). Suppose X satisfies the conditions $(S_1) \sim (S_4)$. Then the following holds.
(a) If every Y_a is discretely θ-expandable, then X is discretely θ-expandable.
(b) If every Y_a is θ-expandable, then X is θ-expandable.

The above theorem can be generalised as follows:

Theorem 17. *Suppose X satisfies conditions $(S_1) \sim (S_3)$ and $(S_5) \sim (S_6)$. Then the following holds.*
(a) If every Y_a is discretely θ-expandable, then X is discretely θ-expandable.
(b) If every Y_a is θ-expandable, then X is θ-expandable.

Proof. (a). Let $\mathcal{F} = \{F_\lambda | \lambda \in \Lambda\}$ be a discrete collection of closed subsets in X. Then $\mathcal{F}_a = \{F_\lambda \cap Y_a | \lambda \in \Lambda\}$ is a discrete collection of closed subsets in Y_a for each $a \in A$. Since Y_a is θ-expandable, there is a sequence $(\mathcal{L}_{a,m})_{m \in \mathbf{N}}$ of collections of open subsets in Y_a such that $\mathcal{L}_{a,m} = \{L_{\lambda,a,m} | \lambda \in \Lambda\}$, satisfying:
$(i)_a$. $F_\lambda \cap Y_a \subset L_{\lambda,a,m}$ for each λ, m.
$(ii)_a$. $L_{\lambda,a,m+1} \subset L_{\lambda,a,m}$ for each λ, m.
$(iii)_a$. For each $y \in Y_a$, there is an $m_y \in \mathbf{N}$ such that $\mathrm{ord}(y, \mathcal{L}_{a,m_y}) < \omega$.

By (S_6), there is a point finite open family $\mathcal{M}_n = \{M_{a,n} | a \in A_n\}$ in X such that $Y_a \setminus X_{n-1} \subset M_{a,n}$ for each $a \in A_n$. Here we may assume that $M_{a,n} \cap X_{n-1} = \emptyset$.

Let us put $H_{\lambda,m} = \bigcup_{n\in\mathbf{N}} \bigcup_{a\in A_n} (p_a^{-1}(L_{\lambda,a,m}) \cap M_{a,n})$ and put $\mathcal{H}_m = \{H_{\lambda,m} | \lambda \in \Lambda\}$. Then $\mathcal{H}_m$ is a collection of open subsets in X for each m and satifies the following conditions:
(1) $F_\lambda \subset H_{\lambda,m}$ for each $\lambda \in \Lambda, m \in \mathbf{N}$.
(2) For each $x \in X$, there is an $m_x \in \mathbf{N}$ such that $\mathrm{ord}(x, \mathcal{H}_{m_x}) < \omega$.

Proof of (1). Let $x \in F_\lambda$. Then, by $(S_1), x \in X_n \setminus X_{n-1}$ for some $n \in \omega$. By $(S_2), x \in Y_a$ for some $a \in A_n$. Then, by $(i)_a, x \in L_{\lambda,a,m}$. Since $Y_a \setminus X_{n-1} \subset M_{a,n}, x \in L_{\lambda,a,m} \cap M_{a,n} \subset H_{\lambda,m}$.

Proof of (2). Let $x \in X$. Then, by $(S_1), x \in X_n \setminus X_{n-1}$ for some $n \in \omega$. Then $x \notin M_{a,l}$ for each $l > n$. Let $A_l^{'} = \{a \in A_l | x \in M_{a,l}\}$ for each $l \leq n$ and put $A^{'} = \bigcup_{i\leq n} A_l^{'}$. Since $\mathcal{M}_l$ is point finite at x for each $l, A^{'}$ is a finite set. Let us put $x_a = p_a(x)$ for each $a \in A$. By $(iii)_a$, there is an $m_a \in \mathbf{N}$ such that $\mathrm{ord}(x_a, \mathcal{L}_{a,m_a}) < \omega$. Let $m^* = \max\{m_a | a \in A'\}$. Then ord $(x, \mathcal{H}_{m^*}) < \omega$.

To show this, let $\Lambda_a = \{\lambda \in \Lambda | x_a \in L_{\lambda,a,m^*}\}$ and put $\Lambda^{'} = \bigcup_{a\in A'} \Lambda_a$. Then, since ord $(x_a, \mathcal{L}_{a,m_a})$ is finite and ord $(x_a, \mathcal{L}_{a,m^*}) <$ ord $(x_a, \mathcal{L}_{a,m_a}), \Lambda_a$ is a finite set. Therefore $\Lambda^{'}$ is a finite set. If $x \in H_{\lambda,m^*}$, then $x \in p_a^{-1}(L_{\lambda,a,m^*}) \cap M_{a,l}$ for some λ and l. Since $x \notin M_{a,l}$ for each $l > n$, we have $l \leq n$. Therefore, if $x \in p_a^{-1}(L_{\lambda,a,m}) \cap M_{a,l}$ for some λ and l, then $a \in A^{'}$. And, since $x_a \in L_{\lambda,a,m^*}, \lambda \in \Lambda_a$.

(b). This proof is quite similar to that of (a). $\square$

Corollary 3. *(a). If every finite subproduct of σ is discretely θ-expandable, then σ is discretely θ-expandable.*
(b)([10]). If every finite subproduct of σ is θ-expandable, then σ is θ-expandable.

Corollary 4. *(a) If every finite subproduct of σ is hereditarily discretely θ-expandable, then σ is hereditarily discretely θ-expandable.*
(b) If every finite subproduct of σ is hereditarily θ-expandable, then σ is hereditarily θ-expandable.

Remark 3. Almost θ-expandability in [10] is the same notion of θ-expandability in [14].

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