MULTIPLE INTEGRALS ON THE SPACE $\Gamma_0(D) \bigoplus M_0(D)$

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Abstract. The space $\Gamma_0(D) \bigoplus M_0(D)$ of generalized functions on an interval D of $\mathbf{R}$ is extended to a susbset D of $\mathbf{R}^2$. We define a translation invariant integral over this space and give some fundamental properties.

1 Introduction The $(E.R)$-integral (Kunugi[1]) preserves the integrability by the translation. The functions with a divergence point of integer power order are not $(E.R)$-integrable. The $(E.R)$-integral was extend to the $(E.R.\nu)$-integral by Okano ([2]), by using an absolutely continuous measure ν in place of Lebesgue measure.

Okano assumed the additional three conditions for a Cauchy sequence to define the $(E.R.\nu)$-integral. One of the conditions asserts that the Cauchy sequence $(V(g_n, \epsilon_n, A_n))$ converges slowly in the sense that there exists an integer k with $k\,\nu(D \setminus A_{n+1}) \geq \nu(D \setminus A_n)$ for $n = 1, 2, ...$ To remove this restriction, we introduced in [3] the $(E.R.\Lambda)$-integral by using a sequence $\Lambda = (\lambda_n)$ of finite absolutely continuous measures. This integral was defined on the space $\Gamma_0(I) \bigoplus M_0(I)$ of generalized functions on an interval I of $\mathbf{R}$. The set $\Gamma_0(I)$ is the singular part of $\Gamma_0(I) \bigoplus M_0(I)$ in the sense that it contains the δ-function and its higher derivatives, and the set $M_0(I)$ consists of all measurable functions on I, which is the regular part of $\Gamma_0(I) \bigoplus M_0(I)$. By a suitable choice of a measure ν (resp. a sequence Λ of measures) , we can find many examples of integrable functions with strong singurality.

However, these integrals are not preserved by the translation. In our previous paper [11], we defined the $(E.R.T)$-integral which is traslationally invariant on the interval I.

In this paper, we define the space $\Gamma_0(D) \bigoplus M_0(D)$ of generalized functions on a subset D of $\mathbf{R}^2$, and extend the $(E.R.T)$-integral to this space.

In Section 2, we define the space $\Gamma_0(D) \bigoplus M_0(D)$ on a subset D of $\mathbf{R}^2$ and define the $(E.R.\Lambda)$-integral over this space.

In Section 3, we extend the $(E.R.T)$-integral to a subset D in $\mathbf{R}^2$.

In Section 4, we introduce two examples of $(E.R.T)$-integrable functions defined on subsets in $\mathbf{R}^2$.

2 The space of generalized functions on $\mathbf{R}^2$and the integral In this section , let D be a closed subset of $\mathbf{R}^2$. The details of the constructions of $\Gamma_0(D) \bigoplus M_0(D)$ on the set D and the integral over the space are omitted, for they are performed in the similar way as the constructions of the space of generalized functions on an interval of $\mathbf{R}$ and the integral in [3].

2.1 The space $\Gamma_0(D) \bigoplus M_0(D)$ on a subset D of $\mathbf{R}^2$ Let $M_0(D)$ be the set of all real valued Lebesgue measurable functions defined on D. In what follows, we suppose that $M_0(D)$ is classified by the usual equivalence relation $f(x) = g(x)$ *a.e.* We denote measurable functions by symbols $f(x), g(x), ...$ and a class in $M_0(D)$ containing a measurable function

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$g(x)$ by the same symbol $g(x)$ or g. For each Lebesgue measurable subset A of D and $\epsilon > 0$, we define a pre-neighbourhood $V(f, \epsilon, A)$ as

$$V(f, \epsilon, A) = \{g \in M_0(D); \iint_A |f(x,y) - g(x,y)| dxdy \leq \epsilon\}.$$

We denote $V(f, \epsilon, A)$ by $V(f)$ if there is no fear of confusion.

A pre-neighbourhood $V(f, \epsilon, A)$ of f is said to be rank n if $m(A) \neq 0$ and $2^{-n} < \epsilon \leq 2^{-n+1}$ for an integer n. The set of pre-neighbourhoods of rank n is denoted by $\mathcal{B}_n$. Moreover, we consider $V(f, \epsilon, A)(= M_0(D))$ with $m(A) = 0$ as a pre-neighbourhood of rank 0, and let $\mathcal{B}_0 = \{M_0(D)\}$. In this way, we are able to introduce a structure of a ranked space in $M_0(D)$.

Definition 1 ([2], p431) *A sequence $(V(f_n))$ of pre-neighbourhoods is called a Cauchy sequence if $V(f_1) \supseteq V(f_2) \supseteq ...$ and $V(f_n) \in \mathcal{B}_{\gamma(n)}$ for some monotone increasing sequence $(\gamma(n)) \in N$ with $\lim_{n\to\infty} \gamma(n) = \infty$.*

We can prove the following lemma similarly as Okano's lemma ([2], p432).

Lemma 1 *If $V(f_n) = V(f_n, \epsilon_n, A_n)$ for $n = 1.2, ...$ and $V(f_1) \supseteq V(f_2) \supseteq ...$, then the following properties hold :*

(1) $\mathrm{m}(A_n \cap (D \setminus A_{n+1})) = 0$ *for every n.*[1]

(2) $\iint_{A_n} |f_n(x,y) - f_{n+1}(x,y)| dxdy \leq \epsilon_n - \epsilon_{n+1}$ *for every n.*

(3) $\sum_{n=k}^{\infty} \iint_{A_k} |f_n(x,y) - f_{n+1}(x,y)| dxdy \leq \epsilon_k$ *for every k.*

The following theorem holds by Lemma 1.

Theorem 1 *If a Cauchy sequence $(V(f_n, \epsilon_n, A_n))$ satisfies the condition such that*

$$\mathrm{m}((D \setminus A_n) \cap ([-1/\epsilon_n, 1/\epsilon_n] \times [-1/\epsilon_n, 1/\epsilon_n])) \leq \epsilon_n \qquad (n = 1, 2, ...),$$

then there exists a function $f \in M_0(D)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in D and $\bigcap_{n=1}^{\infty} V(f_n, \epsilon_n, A_n) = \{f\}$.

For a Cauchy sequence $(V(f_n, \epsilon_n, A_n))$ on D, we consider the following two conditions:

(T_1) $\mathrm{m}((D \setminus A_n) \cap ([-1/\epsilon_n, 1/\epsilon_n] \times [-1/\epsilon_n, 1/\epsilon_n])) \leq \epsilon_n$.

(T_2) f_n is decomposed into a sum of measurable functions f_{1n} and f_{2n} on D, where $\mathrm{supp}(f_{1n}) \subseteq D \setminus A_n$ and

$$\iint_{D \setminus A_n} |f_{2n}(x,y)| dxdy \leq \epsilon_n.$$

If $(V(f_n)) = (V(f_n, \epsilon_n, A_n))$ is a Cauchy sequence which satisfies conditions (T_1) and (T_2) , the Cauchy sequence is called a G_0-Cauchy sequence on D .

Let $\mathbf{H}_0(D)$ be the set of G_0-Cauchy sequences on D , and let $G_0(D)$ be the set of sequences (f_n) in $L^1(D)$ such that there exists a G_0-Cauchy sequence $(V(f_n))$ with $0 \in \bigcap_{n=1}^{\infty} V(f_n)$. [2]

Definition 2 *A decomposition $f_n = f_{1n} + f_{2n}$ in (T_2) is called on associated decomposition of f_n.*

Proposition 1 *If $(f_n), (g_n) \in G_0(D)$, then $(f_n + g_n) \in G_0(D)$. If $(f_n) \in G_0(D)$, then $(\lambda f_n) \in G_0(D)$ for any real number λ.*

[1]The symbol m is the Lebesgue measure.

[2]The set $L^1(D)$ is the set of all Lebesgue integrable functions on D .

If (f_n) and (g_n) have associated decompositions $f_{1n}+f_{2n}$ and $g_{1n}+g_{2n}$ of f_n and g_n respectively such that there is an $n_0 \in \mathbf{N}$ satisfying $f_{1n}=g_{1n}$ *a.e.* for each $n \geq n_0$, we say that (f_n) and (g_n) are equivalent. Let $\Gamma_0(D)$ be the quotient space of $G_0(D)$ classified by this equivalence relation, whose element containing (f_n) is denoted by $[f_n]$.

Example 1 *Let (p_n) be an increasing sequence of real numbers which diverges to ∞ and let (S_n) be a sequence of functions satisfying the following three conditions:*

(i) *S_n is an integrable function on $\mathbf{R}^2$ with*

$$\lim_{n\to\infty}\iint_{\mathbf{R}^2} S_n(x,y)dxdy = 1,$$

(ii) $\sup_{x,y}|S_n(x,y)| \leq p_n,$

(iii) $\mathrm{supp}(S_n) \leq [-1/(2p_n), 1/(2p_n)] \times [-1/(2p_n), 1/(2p_n)]$.

Then the sequence (S_n) is called a δ-type sequence defined by (p_n).

The class $[S_n]$ is denoted by δ. Put $h_n(x,y)=S_n(x-a,y-b)$ for $c=(a,b)\in D$ and each n. Then $[h_n]$ is denoted by δ_c.

Proposition 2 *If $(f_n) \approx (l_n)$ and $(g_n) \approx (k_n)$, then $(f_n+g_n) \approx (l_n+k_n)$ and $(\lambda f_n) \approx (\lambda l_n)$ for any real number λ.*

We set $[f_n]$+$[g_n]$ = $[f_n+g_n]$ and λ $[f_n]$=$[\lambda f_n]$. Hence $\Gamma_0(D)$ turns out to be a linear space. The following set is the underling space of our theory:

$$\Gamma_0(D)\bigoplus M_0(D) = \{([f_n],\ g)\ ;\ [f_n]\in\Gamma_0(D),\ g\in M_0(D)\}.$$

In what follows, we denote the pair $([f_n],\ g)$ by $[f_n]\oplus g$. We will use customary notations in vector space for the addition and the scalar multiplication. The space $\Gamma_0(D)\bigoplus M_0(D)$ is a linear space.

2.2 $(E.R.\Lambda)$**-integration on $\mathbf{R}^2$** Let $\Lambda=(\lambda_n)$ be a sequence of finite measures on $\mathbf{R}^2$ which is absolutely continuous, that is, (1) any Lebesgue measurable set is λ_n-measurable and (2) m$(A)=0$ if and only if $\lambda_n(A)=0$.

Now we introduce a concept of L_0-Cauchy sequence for two dimensional case in the same way as the one dimensional case.

A Cauchy sequence $(V(g_n,\epsilon_n,A_n))$ is called an L_0-Cauchy sequence for Λ if it satisfies the following three conditions on D:

(K_1) if B is a Lebesgue measurable subset of D with $\lambda_n(D\setminus A_n)\geq\lambda_n(B)$, then

$$m(B\cap[-1/\epsilon_n,1/\epsilon_n]\times[-1/\epsilon_n,1/\epsilon_n])\leq\epsilon_n.$$

(K_2) if m$(D\setminus A_n)>0$ for all n, there exist $k,k'>0$ such that

$$k\leq\lambda_n(D\setminus A_n)\leq k'$$

for all n.

(K_3) if B is a Lebesgue measurable subset of D with $\lambda_n(D\setminus A_n)\geq\lambda_n(B)$, then

$$\iint_B |g_n(x,y)|dxdy\leq\epsilon_n.$$

Let $\mathbf{F}_0(\Lambda)$ be the set of L_0-Cauchy sequences on D. A sequence (g_n) with L_0-Cauchy sequence in $\mathbf{F}_0(\Lambda)$ is called an L_0-sequence for Λ . Let $L_0(\Lambda)$ be the set of L_0-sequences (g_n) in $L^1(D)$ for Λ.

Lemma 2 *A sequence* (g_n) *in* $L^1(D)$ *is an element in* $L_0(\Lambda)$ *if* $(g_n)_{n_0}^{\infty}$ *is an element in* $L_0(\Lambda_0)$ *for some* n_0*, where* Λ_0 *is the subsequence* $(\lambda_n)_{n_0}^{\infty}$ *of* $\Lambda = (\lambda_n)$.

Proposition 3 *If* $(g_n), (k_n) \in L_0(\Lambda)$ *, then* $(g_n + k_n) \in L_0(\Lambda)$*. If* $(g_n) \in L_0(\Lambda)$*, then* $(\lambda g_n) \in L_0(\Lambda)$ *for any* $\lambda \in \mathbf{R}$.

Definition 3 *A sequence* $(V(g_n))$ *(resp.* (g_n) *) is called an* L_0*-Cauchy sequence (resp.* L_0*-sequence) for* Λ *and* g*, or for* g *in short, if* $\bigcap_{n=1}^{\infty} V(g_n) = \{g\}$ *for* $(V(g_n)) \in \mathbf{F}_0(\Lambda)$.

We set

$$I_s((g_n);\Lambda) = \limsup_{n\to\infty} \iint_D g_n(x,y)dxdy$$

$$I_i((g_n);\Lambda) = \liminf_{n\to\infty} \iint_D g_n(x,y)dxdy$$

for $(g_n) \in L_0(\Lambda)$.

Theorem 2 *If* (g_n) *and* (f_n) *are* L_0*- sequences for* Λ *and* g*, then*

$$I_s((f_n);\Lambda) = I_s((g_n);\Lambda),$$

$$I_i((f_n);\Lambda) = I_i((g_n);\Lambda).$$

Definition 4 *Let* (g_n) *is an* L_0*-sequence for* Λ *and* g*. If*

$$I_s((g_n);\Lambda) = I_i((g_n);\Lambda),$$

this common value is denoted by

$$I(g,\Lambda) = (E.R.\Lambda) \iint_D g(x,y)dxdy$$

and $I(g,\Lambda)$ *is called the* $(E.R.\Lambda)$*-integral of* g *on* D*. If* $-\infty < I(g,\Lambda) < \infty$*,*
g *is called to be* $(E.R.\Lambda)$*-integrable on* D.

Lemma 3 *Suppose that* $(f_n) \in G_0(D)$ *has a* G_0*-Cauchy sequence* $(V(f_n))$ *with an associated decomposition* $f_{1n} + f_{2n}$ *of* f_n*. Then*

$$\lim_{n\to\infty} \int_D f_{2n}(x)dx = 0.$$

Suppose that a sequence $(f_n) \in G_0(D)$ has an associated decomposition $f_{1n} + f_{2n}$ of f_n such that $\lim_{n\to\infty} \int_D f_{1n}(x)dx$ exists, where the limit value may be finite or infinite. Then by Lemma 3, we have

$$\lim_{n\to\infty} \int_D f_n(x)dx = \lim_{n\to\infty} \int_D f_{1n}(x)dx.$$

Now we give the definition of the $(E.R.\Lambda)$-integral on $\Gamma_0(D) \bigoplus M_0(D)$.

Definition 5 *Suppose that a sequence* (f_n) *in* $G_0(D)$ *has an associated decomposition* $f_{1n} + f_{2n}$ *of* f_n *such that the value*

$$I([f_n];D) = \lim_{n\to\infty} \iint_D f_{1n}(x,y)dxdy$$

exists and the $(E.R.\Lambda)$*-integral* $I(g,\Lambda)$ *of* $g \in M_0(D)$ *exists, where the values of these integrals may be finite or infinite. Then, if* $I([f_n];D) + I(g,\Lambda)$ *has a meaning, this sum is denoted by*

$$(E.R.\Lambda)\iint_D [f_n] \oplus g dxdy = (E.R.\Lambda)\iint_D (f_n(x,y)) \oplus g(x,y)dxdy,$$

and called the $(E.R.\Lambda)$*-integral of* $[f_n]\oplus g$ *on* D*. If* $-\infty < I([f_n];D)+I(g,\Lambda) < \infty$ *,* $[f_n]\oplus g$ *is called to be* $(E.R.\Lambda)$*-integrable on* D*.*

We obtain the linearity of $(E.R.\Lambda)$-integral over the space $\Gamma_0(D) \bigoplus M_0(D)$ excepting the indefinite case.

Example 2 *Put* $D = [0,1]^2$*, and* $G_n = \{(x,y)\ ;\ 1/(2n) \le x \le 1,\ 1/(2n) \le y \le 1\}$*. Let* (λ_n^0) *be a sequence of measures on* D *such that*

$$\lambda_n^0(E) = \begin{cases} \iint_E \exp(-\frac{1}{y})\frac{1}{y^2}dxdy, & on\ [0,1]\times[0,1/(2n)] \\ \iint_E 1dxdy, & on\ G_n \\ \iint_E \exp(-\frac{1}{x})\frac{1}{x^2}dxdy, & on\ [0,1/(2n)]\times[1/(2n),1]. \end{cases}$$

for a measurable subset E *of* D *. We set* $\lambda_n(E) = \lambda_n^0(E)/\exp(-2n)$*, and*

$$f_n(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}, & on\ G_n \\ 0, & on\ \ D\setminus G_n. \end{cases}$$

Then,we find that $(V(f_n,\epsilon_n,G_n))_N^\infty \in \mathbf{F}_0((\lambda_n))$ *for sufficiently large* N*, where* $\epsilon_n = 1/n$*. Moreover, it holds that*

$$(E.R.T((\lambda_n))\iint_D \frac{x^2-y^2}{(x^2+y^2)^2}dxdy = 0.$$

3 A Translation invariant integral on a subset of $\mathbf{R}^2$ In Section 3.1, we recall some terminologies and notations used in [11]. In Section 3.2, the concept of the $(E.R.T)$-integral on an interval of $\mathbf{R}$ is extended to a subset D of $\mathbf{R}^2$ using some terminologies and notations in Section 3.1. We notice that this integral is defined only on the set $M_0(D)$ without considering $\Gamma_0(D)$.

3.1 Terminologies and notations We recall some terminologies and notations used in the definition of the $(E.R.T)$-integral ([11]).

Let I be a finite or infinite open interval in $\mathbf{R}$. We fix two increasing sequences $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ of real numbers with $\lim_{n\to\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\beta_n = \infty$, and a decreasing sequence (J_n) of measurable subsets with $J_n \subseteq [-\beta_n,\beta_n]$ and $\lim_{n\to\infty}m(J_n) = 0$.

Let ν_n be an absolutely continuous measure on $\mathbf{R}$ such that

$$\nu_n(E_n) = \exp(-\alpha_n)$$

for $E_n = \mathbf{R}\setminus[-\beta_n,\beta_n]$ and

$$\nu_n(J_n) = \exp(-\alpha_n)$$

for non empty J_n.

Denote $J_n + a = \{x + a; x \in J_n\}$ by J_n^a. For any measurable subset E of $\mathbf{R}$ and for any different points $a_1, a_2, ..., a_l \in I$, we set

$$\mu_n^0(E) = \sum_{i=1}^{l} \nu_n((E \cap J_n^{a_i}) - a_i) + \nu_n(E \cap E_n) + m(E \cap (CE_n \setminus \bigcup_{i=1}^{l} J_n^{a_i})).^3 \tag{3.1}$$

Let

$$\mu_n = \mu_n^0 / \exp(-\alpha_n) \qquad (n = 1, 2, ...). \tag{3.2}$$

Then (μ_n) is called a sequence of measures defined for $a_1, a_2, ..., a_l$. We denote (μ_n) by $T((a_i)_1^l)$ or $T(a_1, a_2, ..., a_l)$. If $J_{n_0} = \phi$ for some $n_0 \in \mathbf{N}$, the measure μ_n for each $n \geq n_0$ is independent of the choice of points $a_1, a_2, ..., a_l$.

We fix the sequence (ν_n) in the following.

Definition 6 *A sequence (g_n) of functions in $M_0(I)$ is said to satisfy $(*)$-condition for $a_1, a_2, ..., a_l$ if*

$$\lim_{n\to\infty} \int_{J_n^a \cap I} |g_n(x)| dx = 0$$

for any $a \in I$ with $a \neq a_i (i = 1, 2, ..., l)$.

Let $L_0^*(T((a_i)_1^l))$ be the set of all sequences (g_n) in $L_0(T((a_i)_1^l))$ with $(*)$-condition for $a_1, a_2, ..., a_l$.

We define a translation invariant integral over $\Gamma_0(I) \bigoplus M_0(I)$ as follows.

Definition 7 *Let $g \in M_0(I)$ be a function such that , for some sequence $T((a_i)_1^l)$ of measures, there exists a sequence $(g_n) \in L_0^*(T((a_i)_1^l))$ with an L_0-Cauchy sequence $(V(g_n))$ for g. If the $(E.R.T((a_i)_1^l))$-integral of $[f_n] \oplus g$ exists, the $(E.R.T)$-integral of $[f_n] \oplus g$*

$$(E.R.T) \int_D [f_n] \oplus g dx$$

is defined to be the $(E.R.T((a_i)_1^l))$-integral of $[f_n] \oplus g$, where the $(E.R.T)$-integral of $[f_n] \oplus g$ may be finite or infinite. If the $(E.R.T)$-integral of $[f_n] \oplus g$ is finite, $[f_n] \oplus g$ is said to be $(E.R.T)$-integrable.

3.2 A Translation invariant integral Let $(J_n), (E_n), (\beta_n), (\alpha_n)$, and (ν_n) be notations in Section 3.1 and we fix these in the following.

Let P and Q be continuous functions on a finite interval $[a, b]$ with $P \leq Q$. Put

$$D = \{(x, y) \in [a, b] \times \mathbf{R} \ ; \ P(x) \leq y \leq Q(x)\},$$

where $P < Q$ on (a, b). Namely, D is a domain of ordinate type. For any subset A of $\mathbf{R}^2$, we denote

$$(A)_x = A \cap (\mathbf{R}^2)_x,$$

where $(\mathbf{R}^2)_x = \{(x, y) \ ; \ -\infty < y < \infty\}$.

Let $\varphi_1, \varphi_2, ..., \varphi_l$ be continuous functions on $[a, b]$ whose graphs are contained in D. Put

$$I_n{}^{\varphi_i} = \{(x, y) \in [a, b] \times \mathbf{R} \ ; \ y \in J_n + \varphi_i(x)\}. \quad (i = 1, 2, ..., l \ , n = 1, 2...)$$

[3] $CE_n = R \setminus E_n$

For each x, we consider a measure in Section 3.1 on a parallel line to the y-axis which goes through the point $(x, 0)$. For each subset E of $(\mathbf{R}^2)_x$,

$$\text{(3.3)} \quad \mu^0_{n,x}(E) = \sum_{i=1}^{l}{}' \nu_n((E \cap (I_n{}^{\varphi_i})_x) - \varphi_i(x)) + \nu_n(E \cap (R \times E_n)_x) +$$

$$+ m(E \cap (C(R \times E_n)_x \setminus \bigcup_{i=1}^{l} (I_n{}^{\varphi_i})_x)),$$

where $(I_n{}^{\varphi_i})_x = \phi$ for x in $R \setminus [a, b]$. Here, the symbol $\sum'$ means that the summation is taken only for the different values in $\{\varphi_i(x) \ ; \ i = 1, 2, .., l\}$.

Put

$$\text{(3.4)} \qquad \mu_{n,x} = \frac{\mu^0_{n,x}}{\exp(-\alpha_n)} \qquad (n = 1, 2, ...).$$

Namely, we have $(\mu_{n,x}) = T((\varphi_i(x))^l_1)$.

Let τ^0_n be a measure on $\mathbf{R}^2$ defined by

$$\text{(3.5)} \qquad \tau^0_n(F) = \int_{-\beta_n}^{\beta_n} \mu^0_{n,x}((F)_x) dx + \sigma_n((E_n \times \mathbf{R}) \cap F)$$

for $F \subseteq \mathbf{R}^2$, where σ_n is an absolutely continuous measure on $E_n \times \mathbf{R}$ with $\sigma_n(E_n \times \mathbf{R}) = \exp(-\alpha_n)$. Put

$$\text{(3.6)} \qquad \tau_n = \frac{\tau^0_n}{\exp(-\alpha_n)} \qquad (n = 1, 2, ...).$$

Then, (τ_n) is called a sequence of measures defined for $\varphi_1, \varphi_2, ..., \varphi_l$. We denote (τ_n) by $T((\varphi_i)^l_1)$ or $T(\varphi_1, \varphi_2, ..., \varphi_l)$.

We fix sequence (σ_n) in the following.

We define a translation invariant integral on D by using a sequence $T((\varphi_i)^l_1)$ of measures.

Suppose that $(V(f_n(x, \cdot))) = (V(f_n(x, \cdot), \epsilon_n, (G_n)_x)) \in \mathbf{F}_0(T((\varphi_i(x))^l_1))$ for almost all x. Then, for almost all x, $V(f_n(x, \cdot))$ satisfies (K_2)-condition. Namely, putting $T((\varphi_i(x))^l_1) = (\mu_{n,x})$, if m$((D \setminus G_n)_x) > 0$ for all n, there exist positive constants c, c' such that

$$\text{(3.7)} \qquad c' \leq \mu_{n,x}((D \setminus G_n)_x) \leq c$$

for all $n \in \mathbf{N}$. A Cauchy sequence $(V(f_n(x, \cdot)))$ is said to satisfy (K_2)-condition uniformly in x if c and c' are independent of x.

Theorem 3 *Assume that a sequence (f_n) on D satisfies the following two conditions:*

(i) *For almost all x, there exists an L_0-Cauchy sequence $(V(f_n(x, \cdot))) = (V(f_n(x, \cdot), \epsilon_n, (G_n)_x))$ for $f(x, \cdot)$ and $T((\varphi_i(x))^l_1)$, where ϵ_n is independent of x for each n and $(V(f_n(x, \cdot)))$ satisfies the (K_2)-condition uniformly in x.*

(ii) *$|f_n| \leq r_n$ on a.e.in D $(n = 1, 2, ...)$ for an increasing divergent sequence (r_n) such that $(r_n \exp(-\alpha_n))^\infty_{n_0}$ is a monotone decreasing sequence for some n_0 which converges to 0.*

Then, there exists an L_0-Cauchy sequence in $\mathbf{F}_0(T((\varphi_i)^l_1))$ for f.

Proof. Put $(\mu_{n,x}) = T((\varphi_i(x))^l_1)$ and $(\tau_n) = (T((\varphi_i)^l_1))$. Let (g_n) be a sequence on D such taht

$$g_n(x,y) = \begin{cases} f_n(x,y), & \text{on } B_n \\ 0, & \text{on } D \setminus B_n, \end{cases}$$

where $B_n = (D \setminus W_n) \cap G_n$ for $W_n = \bigcup_{i=1}^{l} I_n{}^{\varphi_i}$.

Since $(V(f_n(x,\cdot)))$ satisfies (K_2)-condition uniformly in x, there exist positive constants c, c' such that

$$c' \leq \mu_{n,x}((D \setminus G_n)_x) \leq c \tag{3.8}$$

for all x.

We will show that $(V(g_n))_N^\infty = (V(g_n, \eta_n, B_n))_N^\infty \in \mathbf{F}_0(T((\varphi_i)_1^l))$ for sufficiently large $N \in \mathbf{N}$ with $n_0 \leq N$, where $\eta_n = (l+c)(b-a+1)(\omega_n + \epsilon_n + m(J_n))$ for $\omega_n = r_n \exp(-\alpha_n)$.

By (3.8), we have

$$c'(b-a) \leq \tau_n(D \setminus G_n) \leq c(b-a). \tag{3.9}$$

Hence $(V(g_n))$ satisfies (K_2)-condition. By (3.9) , we see that

$$\tau_n^0(D \setminus G_n) \leq c(b-a)\exp(-\alpha_n). \tag{3.10}$$

Moreover, we have

$$\tau_n^0(W_n) \leq \int_a^b \sum_{i=1}^{l} \nu_n((I_n{}^{\varphi_i})_x)dx = l(b-a)\exp(-\alpha_n). \tag{3.11}$$

By virtue of (3.10) and (3.11), we find

$$\tau_n^0(D \setminus B_n) \leq (c+l)(b-a)\exp(-\alpha_n). \tag{3.12}$$

Let B be a subset of D such that $\tau_n^0(D \setminus B_n) \geq \tau_n^0(B)$. Then, by (3.12), we have

$$\begin{aligned} (c+l)(b-a)\exp(-\alpha_n) &\geq \tau_n^0(D \setminus B_n) \geq \tau_n^0(B) \\ &\geq \tau_n^0(B \cap B_n) = m(B \cap B_n) \quad . \end{aligned} \tag{3.13}$$

Hence, by (3.13), it holds that

$$\begin{aligned} \iint_B |g_n(x,y)|dxdy &= \iint_{B \cap B_n} |f_n(x,y)|dxdy \\ &\leq r_n(c+l)(b-a)\exp(-\alpha_n) \leq \eta_n. \end{aligned}$$

Thus $(V(g_n))_N^\infty$ satisfies (K_3)-condition.

Next, we will show that $(V(g_n))_N^\infty$ satisfies (K_1)-condition for (τ_n). For any subset B of D with $\tau_n^0(D \setminus B_n) \geq \tau_n^0(B)$, we find that, by (3.13),

$$\begin{aligned} m(B \cap (D \setminus W_n)) &= \int_a^b m((B \cap (D \setminus W_n))_x)dx \\ &= \tau_n^0(B \cap (D \setminus W_n)) \leq \tau_n^0(B) \leq (c+l)(b-a)\exp(-\alpha_n). \end{aligned} \tag{3.14}$$

Moreover, we have

$$\mathrm{m}(B \cap W_n) \leq \mathrm{m}(W_n) \leq l(b-a)\mathrm{m}(J_n) \tag{3.15}$$

Therefore, by (3.14) and (3.15), we obtain

$$m(B \cap [-1/\eta_n, 1/\eta_n] \times [-1/\eta_n, 1/\eta_n]) \leq m(B) \leq \eta_n$$

for sufficiently large n. Thus $(V(g_n))_N^\infty$ satisfies (K_1).

Moreover, since $(V(f_n(x,\cdot)))$ is a Cauchy sequence for almost all x, we have

$$\int_{(G_n)_x} |f_n(x,y) - f_{n+1}(x,y)|dy \le \epsilon_n - \epsilon_{n+1}.$$

Hence, it holds that

$$\begin{aligned} &\iint_{B_n} |g_n(x,y) - g_{n+1}(x,y)|dxdy \\ &\le \int_a^b \int_{(G_n)_x} |f_n(x,y) - f_{n+1}(x,y)|dydx \le \eta_n - \eta_{n+1}, \end{aligned}$$

so that $(V(g_n))_N^\infty$ is a Cauchy sequence. This completes the proof.

Theorem 4 *Assume that a sequence* (f_n) *on* D *satisfies conditions* (i) and (ii) *in* Theorem 3 *and the integral*

$$\lim_{n\to\infty} \int_a^b F_n(x)dx$$

exists in the sense that the limit is finite or infnite, where F_n *is a function on* $[a,b]$ *defined by*

$$F_n(x) = \int_{P(x)}^{Q(x)} f_n(x,y)dy.$$

Then $(E.R.T((\varphi_i)_1^l))$*-integral of* f *exists on* D*, and*

$$(E.R.T((\varphi_i)_1^l)) \iint_D f(x,y)dxdy = \lim_{n\to\infty} \int_a^b F_n(x)dx.$$

Proof. We use the notations in the proof of Theorem 3. We have

$$\begin{aligned} &|\iint_{G_n} f_n(x,y)dxdy - \iint_{B_n} g_n(x,y)dxdy| \\ &\le \iint_{G_n\setminus B_n} |f_n(x,y)|dxdy \le \int_a^b \int_{(W_n)_x} |f_n(x,y)|dydx \end{aligned} \tag{3.16}$$

Since $(V(f_n(x,\cdot)))$ satisfies (K_2)-condition uniformly in x , there exist positive constants c, c' such that

$$c' \le \mu_{n,x}((D \setminus G_n)_x) = \frac{\mu^0_{n,x}((D \setminus G_n)_x)}{\exp(-\alpha_n)} \le c$$

for all x. Hence, it follows that

$$\mu^0_{n,x}((W_n)_x) \le l \exp(-\alpha_n) \le l\ \mu^0_{n,x}((D \setminus G_n)_x)/c'.$$

Let k be an integer with $l/c' \le k$. By virtue of (K_3)-condition , we have

$$\int_{(W_n)_x} |f_n(x,y)|dy \le k\ \epsilon_n. \tag{3.17}$$

Here, ϵ_n is independent of x. Therefore, by (3.16) and (3.17), we obtain

$$\lim_{n\to\infty}\iint_{G_n} f_n(x,y)dxdy = \lim_{n\to\infty}\iint_{B_n} g_n(x,y)dxdy. \tag{3.18}$$

On the other hand, (K_3)-condition implies

$$\begin{aligned}|\int_a^b \int_{P(x)}^{Q(x)} f_n(x,y)dy - \int_a^b \int_{(G_n)_x} f_n(x,y)dydx| \\ \leq \int_a^b \int_{(D\setminus G_n)_x} |f_n(x,y)|dydx \leq (b-a)\,\epsilon_n.\end{aligned} \tag{3.19}$$

Hence, we have , by (3.18) and (3.19) ,

$$\begin{aligned}\lim_{n\to\infty}\int_a^b F_n(x)dx &= \lim_{n\to\infty}\int_a^b \int_{(G_n)_x} f_n(x,y)dydx \\ &= \lim_{n\to\infty}\iint_{B_n} g_n(x,y)dxdy.\end{aligned}$$

Since $(V(g_n))$ is an L_0-Cauchy sequence and

$$\lim_{n\to\infty}\iint_{B_n} g_n(x,y)dxdy$$

exists, $(E.R.T((\varphi_i)_1^l))$-integral of f exists on D and

$$(E.R.T((\varphi_i)_1^l))\iint_D f(x,y)dxdy = \lim_{n\to\infty}\int_a^b F_n(x)dx.$$

Thus we obtain the assertion.

Definition 8 *Let (f_n) be a sequence of functions in $L^1(D)$ satisfying the following four conditions:*

$(\mathcal{O}_1)$ *For almost all x, $(f_n(x,\cdot))$ satisfies $(*)$-condition for $\varphi_1(x), \varphi_2(x), ..., \varphi_l(x)$ and there exists a sequence $(V(f_n(x,\cdot))) = (V(f_n(x,\cdot), \epsilon_n, (G_n)_x))$ in $\mathbf{F}_0(T((\varphi_i(x))_1^l))$ on $(D)_x$, where $(V(f_n(x,\cdot)))$ satisfies (K_2)-condition uniformly in x and ϵ_n is independent of x for each n.*

$(\mathcal{O}_2)$ *There exists a finite number of points $a_1, a_2, .., a_m \in [a,b]$ such that $(F_n) \in L_0^*(T((a_i)_1^m))$,where*

$$F_n(x) = \int_{P(x)}^{Q(x)} f_n(x,y)dy.$$

$(\mathcal{O}_3)$ *The following limit*

$$\lim_{n\to\infty}\int_a^b \int_{P(x)}^{Q(x)} f_n(x,y)dydx$$

exists, where the limit may be finite or infinite.

$(\mathcal{AO})$ *There exists an increasing divergent sequence (r_n) such that (a) the sequence $(r_n \exp(-\alpha_n))$ converges monotonically to 0 for sufficiently large n, and (b) $|f_n| \leq r_n$ a.e. on D $(n = 1, 2, ...)$.*

Let $\mathcal{O}(D; T((\varphi_i)_1^l))$ be the set of all sequences (f_n) in $L^1(D)$ satisfying $(\mathcal{O}_1)$, $(\mathcal{O}_2)$, $(\mathcal{O}_3)$, and $(\mathcal{AO})$. A sequence $(f_n) \in \mathcal{O}(D; T((\varphi_i)_1^l))$ is called an $\mathcal{O}$-sequence for $T((\varphi_i)_i^l)$, and the $\mathcal{O}$-sequence is called a sequence related to f if $\bigcap_{n=1}^\infty V(f_n(x,\cdot)) = \{f(x,\cdot)\}$ for a.a.x.

Let $\mathcal{T}(\mathbf{R}^2)$ be the set of all sequences $T((\varphi_i)_1^m)$ of measures. The set $\mathcal{T}(\mathbf{R}^2)$ is an ordered set with respect to the order $T((\varphi_i)_1^l) \leq T((\psi_i)_1^p)$ defined by $\{\varphi_1, \varphi_2, ..., \varphi_l\} \subseteq \{\psi_1, \psi_2, ..., \psi_p\}$.

Theorem 5 *Let* (f_n) *and* (g_n) *be* $\mathcal{O}$*-sequences in* $\mathcal{O}(D; T((\varphi_i)_1^l))$ *and* $\mathcal{O}(D; T((\psi_i)_1^p))$ *respectvely related to* f*. If* $\{\psi_1, \psi_2, ..., \psi_p\}$ *contains* $\{\varphi_1, \varphi_2, .. ., \varphi_l\}$*, then*

$$(E.R.(T((\varphi_i)_1^l))) \iint_D f(x,y)dxdy = (E.R.T((\psi_i)_1^p)) \iint_D f(x,y)dxdy.$$

Proof. By the assumption of this theorem, there exist L_0-Cauchy sequences $(V(f_n(x,\cdot)))$, $(V(g_n(x,\cdot)))$ for almost all x such that

$$(3.20) \qquad \bigcap_{n=1}^{\infty} V(f_n(x,\cdot)) = \bigcap_{n=1}^{\infty} V(g_n(x,\cdot)) = \{f(x,\cdot)\} \ a.e.x.$$

Putting $F_n(x) = \int_{P(x)}^{Q(x)} f_n(x,y)dy$ and $G_n(x) = \int_{P(x)}^{Q(x)} g_n(x,y)dy$, there exist L_0-Cauchy sequences $(V(F_n))$ and $(V(G_n))$ by $(\mathcal{O}_2)$-condition. Hence, by virtue of Theorem 1, the both limits $\lim_{n\to\infty} F_n(x)$ and $\lim_{n\to\infty} G_n(x)$ exist almost everywhere. Therefore, since the integral of $f(x,\cdot)$ on $(D)_x$ exists uniquely by (3.20), and Proposition 2 in [11], we have

$$(3.21) \qquad \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} G_n(x) \quad a.e.$$

Moreover, by $(\mathcal{O}_2)$-condition, there exist two finite sets $\{a_1, a_2, .., a_r\}$ and $\{b_1, b_2, ..., b_q\}$ such that $(F_n) \in L_0^*(T((a_i)_1^r))$ and $(G_n) \in L_0^*(T((b_i)_1^q))$. Let $\{c_1, c_2, ..., c_e\}$ be the union of $\{a_1, a_2, .., a_r\}$ and $\{b_1, b_2, ..., b_q\}$. By virtue of Proposition 1 in [11], we have $(F_n), (G_n) \in L_0^*(T((c_i)_1^e))$. Hence, according to (3.21), $(\mathcal{O}_3)$-condition and Proposition 2 in [11], we have

$$\lim_{n\to\infty} \int_a^b F_n(x)dx = \lim_{n\to\infty} \int_a^b G_n(x)dx,$$

so that we have the assertion by Theorem 4.

By the symmetry of arguments, we can change the role of x and y in the above discussion. Let R and S be continuous functions on $[c,d]$ with $R \leq S$, where $R < S$ on (c,d). Put

$$D = \{(x,y) \in \mathcal{R} \times [c,d] \ ; \ R(y) \leq x \leq S(y)\}.$$

Namely, D is a domain of abscissa type. Let $\phi_1, \phi_2, .., \phi_m$ be continuous functions on $[c,d]$ such that $R \leq \phi_i \leq S \quad (i = 1, 2, ..., m)$ on $[c,d]$.

Let ϑ_n be a measure on $\mathbf{R} \times E_n$ with $\vartheta_n(\mathbf{R} \times E_n) = \exp(-\alpha_n)$. Using the similar equations as (3.3),(3.4),(3.5),and(3.6), we define a measure

$$(3.22) \quad \rho_n(F) = \int_{-\beta_n}^{\beta_n} d_{n,y}((F)_y)dy + \vartheta_n((\mathbf{R} \times E_n) \cap F)$$

for each subset F of $\mathbf{R}^2$, where $(F)_y = \{(x,y) \ ; \ x \in F\}$ and

$$(3.23) \qquad (d_{n,y}) = T((\phi_i(y))_1^m).$$

We shall call (ρ_n) a sequence of measures defined for $\phi_1, \phi_2, .., \phi_m$, and denote it by $T((\phi_i)_1^m)$ or $T(\phi_1, \phi_2, .., \phi_m)$.

Let $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, and $(\mathcal{A}_3)$ be the conditions corresponding to $(\mathcal{O}_1)$, $(\mathcal{O}_2)$, and $(\mathcal{O}_3)$ when x is replaced by y. Let $\mathcal{A}(D;T((\phi_i)_1^m))$ be the set of all sequences (f_n) in $L^1(D)$ satisfying $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ and $(\mathcal{AO})$. A sequence $(f_n) \in \mathcal{A}(D;\ T((\phi_i)_1^m))$ is called a $\mathcal{A}$-sequence for $T((\phi_i)_1^m)$, and the $\mathcal{A}$-sequence is called a sequence related to f if $\bigcap_{n=1}^{\infty} V(f_n(\cdot,y)) = \{f(\cdot,y)\}$ for $a.a.y$.

Let (f_n) and (g_n) be $\mathcal{A}$-sequences in $\mathcal{A}(D;T(\phi_i)_1^m))$ and $\mathcal{A}(D;T(\theta_i)_1^k))$ related to f respectively. Then, in the same way as Theorem 5, we find that ,if $\{\phi_1,\phi_2,...,\phi_m\} \supseteq \{\theta_1,\theta_2,...,\theta_k\}$,

$$(E.R.T((\phi_i)_1^l)) \iint_D f(x,y)dxdy = (E.R.T((\theta_i)_1^k) \iint_D f(x,y)dxdy.$$

Definition 9 *Assume that there exist an $\mathcal{O}$-sequence (f_n) in $\mathcal{O}(D;T((\varphi_i)_1^l))$ (resp. an $\mathcal{A}$-sequence (f_n) in $\mathcal{A}(D;T((\phi_i)_1^m))$ related to f. Then we denote the integral $(E.R.T((\varphi_i)_1^l)) \iint_D f(x,y)dxdy$ (resp.$(E.R.T((\phi_i)_1^m)) \iint_D f(x,y)dxdy$) by*

$$(E.R.T)_{\mathcal{O}} \iint_D f(x,y)dxdy$$

$$(resp.\ (E.R.T)_{\mathcal{A}} \iint_D f(x,y)dxdy).$$

If the integral is finite, f is said to be $(E.R.T)_{\mathcal{O}}$-integrable (resp. $(E.R.T)_{\mathcal{A}}$-integrable) on D.

The $(E.R.T)_{\mathcal{O}}$-integral and $(E.R.T)_{\mathcal{A}}$-integral are invariant under the translation.

For an $\mathcal{O}$-sequence $(f_n) \in \mathcal{O}(D;T((\varphi_i)_1^l))$, we see that

$$(E.R.T)_{\mathcal{O}} \iint_D f(x,y)dxdy = (E.R.T)\int_a^b (E.R.T)\int_{P(x)}^{Q(x)} f(x,y)dydx,$$

where $D = \{(x,y) \in [a,b] \times \mathbf{R};\ P(x) \le y \le Q(x)\}$. Moreover, for an $\mathcal{A}$-sequence (f_n) in $\mathcal{A}(D;T((\phi_i)_1^m))$, we see that

$$(E.R.T)_{\mathcal{A}} \iint_D f(x,y)dxdy = (E.R.T)\int_c^d (E.R.T)\int_{R(y)}^{S(y)} f(x,y)dxdy,$$

where $D = \{(x,y) \in \mathbf{R} \times [c,d];\ R(y) \le y \le S(y)\}$.

Let D be a domain of ordinate type as well as abscissa type. Then the following corollary holds.

Proposition 4 *If (f_n) is a sequence in the intersection of $\mathcal{O}(D;T((\varphi_i)_1^l))$ and $\mathcal{A}(D;T((\phi_i)_1^m))$ related to f, then*

$$(E.R.T)_{\mathcal{O}} \iint_D f(x,y)dxdy = (E.R.T)_{\mathcal{A}} \iint_D f(x,y)dxdy. \tag{3.24}$$

Proof. It holds that ,by Theorem 4,

$$(E.R.T)_{\mathcal{O}} \iint_D f(x,y)dxdy = \lim_{n\to\infty} \iint_D f_n(x,y)dxdy$$

$$= (E.R.T)_{\mathcal{A}} \iint_D f(x,y)dxdy.$$

The comman value in (3.24) is denoted by

$$(E.R.T) \iint_D f(x,y)dxdy.$$

4 Some examples of integrable functions We discuss on two examples of integrable functions.

Example 1 *Let D be the set $[0,1]^2$. Let (U_n) be a sequence of subsets of D defined inductively by*

$$U_0 = ((\frac{1}{2},\ 1) \times (0,\ \frac{1}{2})) \cup ((0,\ \frac{1}{2}) \times (\frac{1}{2},\ 1))$$

and

$$U_{n+1} = \frac{1}{2}\{U_n \cup (U_n + (1,1))\} \qquad n \in \mathcal{N}$$

Then we have a fractal set $\bigcup_{n=0}^{\infty} D \setminus U_n$. Let f be a function on D defined by

$$f(x,y) = \begin{cases} (-1)^n \frac{2^{n+1}}{n+1}, & \text{on } U_n \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we set

$$f_n(x,y) = \begin{cases} f(x,y), & \text{on } G_n \\ 0, & \text{otherwise,} \end{cases}$$

where $G_n = \overline{\bigcup_{\nu=0}^{2n} U_\nu}$.

We set $\alpha_n = n$ log4 and $J_n = \phi$. Then we have

$$\mu^0_{n,x}(E) = \mathrm{m}(E)$$

for any subset E of $(D)_x$, and $\mu^0_{n,x} = T(\varphi_1(x)))$ for φ_1 vanishing on $[0,1]$. We can see that $(V(f(x,\cdot),1/n,(G_n)_x)) \in \mathbf{F}_0(T((\varphi_1(x)))$ for any $x \in [0,1]$. Hence (f_n) satisfies $(\mathcal{O}_1)$-condition.

Let F_n be a function on $[0,1]$ defined by

$$F_n(x) = \int_{(G_n)_x} f_n(x,y)dy.$$

Then we have $(V(F_n.1/n, A_n)) \in \mathbf{F}_0(T(a_1))$, where $a_1 = 0$ and $A_n = [0,1]$. Indeed,

$$\int_{A_n} |F_n(x) - F_{n+1}(x)|dx = |\sum_{\nu=2n+1}^{2n+2} (-1)^\nu \frac{1}{n+1}|$$

$$= \frac{1}{2n+2} - \frac{1}{2n+3}.$$

Hence $(V(F_n, 1/n, A_n))$ is a Cauchy sequence. Hence (f_n) satisfies $(\mathcal{O}_2)$-condition. Letting $r_n = 2^{2n+1}/(2n+1)$, we obtain $|f_n| \le r_n$ and the sequence $(r_n\exp(-\alpha_n))$ converges monotonically to 0. Hence (f_n) satisfies $(\mathcal{AO})$-condition. Moreover, we have

$$\lim_{n\to\infty} \int_0^1 F_n(x)dx = \lim_{n\to\infty} \sum_{\nu=1}^{2n} (-1)^\nu \frac{1}{\nu+1} = \log 2.$$

Hence (f_n) *satisfies* $\mathcal{O}_3$*-condition. Thus,* $(f_n) \in \mathcal{O}(D; T(\varphi_1))$.

By a similar argument, we find that $(f_n) \in \mathcal{A}(D; T(\varphi_1))$. *Hence we obtain*

$$(E.R.T) \iint_D f(x,y)dxdy = (E.R.T) \int_{-1}^{1} (E.R.T) \int_{-1}^{1} f(x,y)dydx$$
$$= (E.R.T) \int_{-1}^{1} (E.R.T) \int_{-1}^{1} f(x,y)dxdy = \log 2.$$

Example 2 *Let* D *and* J_n *be sets* $[-1,1]^2$ *and* $[-1/(2n), 1/(2n)]$ *respectively. Let* ν_n *be a measure defined by*

$$\nu_n(E) = \int_E \exp(-\frac{1}{|x|}) \ \frac{1}{x^2} dx$$

for any measurable subset E *of* J_n. *For* $|c| < 1$, *we set*

$$f_n(x,y) = \begin{cases} \frac{1}{x-y+c}, & \text{on } G_n \\ 0, & \text{otherwise,} \end{cases}$$

where G_n *is the set* $\{(x,y) \in D;\ |y - \varphi_1(x)| > 1/(2n)\}$ *given by a function* $\varphi_1(x) = x + c$. *We can show that* $(f_n) \in \mathcal{O}(D; T(\varphi_1))$. *Indeed, we obtain* $(V(f_n(x,\cdot), 1/n, (G_n)_x)_N^\infty \in \mathbf{F}_0(T(\varphi_1(x)))$ *for a sufficiently large* $N \in \mathbf{N}$ *uniformly in* x. *Let* F_n *be a function defined by*

$$F_n(x) = \begin{cases} \int_{(G_n)_x} f_n(x,y)dy, & \text{on } B_n \\ 0, & \text{otherwise,} \end{cases}$$

where

$$B_n = [-1,1] \setminus \{x;\ |x+1+c| < \frac{1}{2n},\ |x-1+c| < \frac{1}{2n}\}.$$

Then we have

$$F_n(x) = \log|x+1+c| - \log|x-1+c|$$

on B_n, *and* $(V(F_n, 1/n, B_n))_{N'}^\infty \in \mathbf{F}_0(T(a_1.a_2))$ *for a sufficiently large* N', *where* $a_1 = -1-c$ *and* $a_2 = 1-c$. *Moreover, we obtain*

$$lim_{n\to\infty} \int_{B_n} F_n(x)dx = \begin{cases} c\ \log(\frac{4}{c^2} - 1) + 2\log\frac{2+c}{2-c}, & c \neq 0 \\ 0, & c = 0. \end{cases}$$

It is easy to show that (f_n) *satisfies the remaining conditions.*

Similarly as the above argument, we have $(f_n) \in \mathcal{A}(D; T((\phi_1))$, *where* $\phi_1(y) = y - c$ *on* $-1 \ \leq \ y \leq 1$. *Therefore, it holds that*

$$(E.R.T) \iint_D \frac{1}{x-y+c} dxdy = (E.R.T) \int_{-1}^{1} (E.R.T) \int_{-1}^{1} \frac{1}{x-y+c} dydx$$
$$= (E.R.T) \int_{-1}^{1} (E.R.T) \int_{-1}^{1} \frac{1}{x-y+c} dxdy.$$

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