EXPANSIONS OF SUBALGEBRAS AND IDEALS IN BCK/BCI-ALGEBRAS

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ABSTRACT. The notions of an expansion of subalgebras (resp., ideals), σ-primary ideals, and residual divisions are introduced, and relate d properties are investigated.

1. INTRODUCTION

The notion of BCK-algebras was proposed by Imai and Iséki in 1966. In the same year, Iséki introduced the notion of BCI-algebras which is a generalization of BCK-algebras. For the general development of BCK/BCI-algebras, the ideal theory plays an important role. In this paper, we introduce the notion of expansions of subalgebras and ideals in BCK/BCI-algebras, and the notion of σ-primary ideals in BCK-algebras. We also define the notion of residual division, and investigates related properties.

2. PRELIMINARIES

We give herein the basic notions on BCK/BCI-algebras. For further information, we refer the reader to the book [4]. By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,
(ii) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,
(iii) $(\forall x \in X)\,(x * x = 0)$,
(iv) $(\forall x, y \in X)\,(x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. In a BCI-algebra X, the following hold:

(z1) $(\forall x \in X)\,(x * 0 = x)$,
(z2) $(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)$,
(z3) $(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)$,
(z4) $(\forall x, y \in X)\,(0 * (x * y) = (0 * x) * (0 * y))$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. A BCK-algebra X is said to be *commutative* if it satisfies the equality:

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * x)). \tag{1}$$

Note that a BCI-algebra satisfying the equality (1) is a BCK-algebra (see [3]). In what follows let X denote a BCK/BCI-algebra unless otherwise specified. A nonempty subset A of X is called a *subalgebra* of X if $x * y \in A$ for all $x, y \in A$. A nonempty subset A of X is called an *ideal* of X if it satisfies

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- $0 \in A$,
- $(\forall x \in X)(\forall y \in A)\,(x * y \in A \Rightarrow x \in A)$.

Note that if x is an element of an ideal A of X and $y \leq x$, then $y \in A$. An ideal A of a BCI-algebra X is said to be *closed* if $0*x \in A$ whenever $x \in A$. Note that every closed ideal (resp., ideal) of a BCI-algebra (resp. BCK-algebra) X is a subalgebra of X. A proper ideal I of a commutative BCK-algebra X is said to be *prime* if it satisfies:

$$(\forall x, y \in X)\,(x \wedge y \in I \Rightarrow x \in I \text{ or } y \in I),$$

where $x \wedge y = y*(y*x)$. For any elements $x, y \in X$, let us write $x*y^k$ for $(\cdots((x*y)*y)*\cdots)*y$ in which y occurs k times.

For a positive integer k, the *k-nil radical* (see [2]) of a subset G of a BCI-algebra X is defined to be the set of all elements of X satisfying $0 * x^k \in G$, denoted by $\sqrt[k]{G}$, i.e.,

$$\sqrt[k]{G} := \{x \in X \mid 0 * x^k \in G\}.$$

Note that $\sqrt[k]{G}$ does not contain G itself in general (see [2]).

3. Expansions of subalgebras and ideals

Definition 3.1. Let $\mathbb{O}(X)$ be a set of objects in X. An *expansion of objects* in X is defined to be a function $\sigma : \mathbb{O}(X) \to \mathbb{O}(X)$ such that

(o1) $(\forall G \in \mathbb{O}(X))\,(G \subseteq \sigma(G))$.
(o2) $(\forall G, H \in \mathbb{O}(X))\,(G \subseteq H \Rightarrow \sigma(G) \subseteq \sigma(H))$.

Let $\mathbb{S}(X)$ (resp., $\mathbb{I}(X)$) denote the set of all subalgebras (resp., ideals) of X. If $\mathbb{O}(X) = \mathbb{S}(X)$ (resp., $\mathbb{O}(X) = \mathbb{I}(X)$), we say that σ is an *expansion of subalgebras* (resp., *ideals*).

Lemma 3.2. [2] *Let X be a BCI-algebra. If $G \in \mathbb{S}(X)$, then $G \subseteq \sqrt[k]{G}$ for every positive integer k.*

Lemma 3.3. [2] *Let X be a BCI-algebra. For every subsets G and H of X, if $G \subseteq H$ then $\sqrt[k]{G} \subseteq \sqrt[k]{H}$ for every positive integer k.*

Example 3.4. (1) The function $\sigma_0 : \mathbb{S}(X) \to \mathbb{S}(X)$ (resp., $\sigma_0 : \mathbb{I}(X) \to \mathbb{I}(X)$) defined by $\sigma_0(G) = G$ for all $G \in \mathbb{S}(X)$ (resp., $\mathbb{I}(X)$) is an expansion of subalgebras in X.

(2) The function ν that assigns the largest subalgebra (resp., ideal) X to each subalgebra (resp., ideal) of X is an expansion of subalgebras (resp., ideals) in X.

(3) For each ideal I of X, let

$$\mathfrak{M}(I) = \cap\{M \mid I \subseteq M,\ M \text{ is a maximal ideal of } X\}.$$

Then $\mathfrak{M}$ is an expansion of ideals in X.

(4) Let X be a BCI-algebra and let $\sigma_k : \mathbb{S}(X) \to \mathbb{S}(X)$ be defined by $\sigma_k(G) = \sqrt[k]{G}$ for all $G \in \mathbb{S}(X)$. Then σ_k is an expansion of subalgebras in X, where k is a positive integer.

(5) Let X be a commutative BCK-algebra and let $I \in \mathbb{I}(X)$. For each $a \in X$, the set $a^{-1}I := \{x \in X \mid a \wedge x \in I\}$ is an ideal of X containing I, and if I and J are ideals of X such that $I \subseteq J$ then $a^{-1}I \subseteq a^{-1}J$ (see [1]). Hence the function $\sigma_a : \mathbb{I}(X) \to \mathbb{I}(X)$ given by $\sigma_a(I) = a^{-1}I$ for all $I \in \mathbb{I}(X)$ is an expansion of ideals in X.

Definition 3.5. Let σ be an expansion of ideals in a commutative BCK-algebra X. Then an ideal G of X is said to be *σ-primary* if

$$(\forall a, b \in X)\,(a \wedge b \in G,\ a \notin G \Rightarrow b \in \sigma(G)).$$

Note that an ideal G of a commutative BCK-algebra X is σ_0-primary if and only if it is a prime ideal of X, where σ_0 is the function in Example 3.4(1).

Theorem 3.6. *Let X be a commutative BCK-algebra. If σ and δ are expansions of ideals in X such that $\sigma(G) \subseteq \delta(G)$ for every $G \in \mathbb{I}(X)$, then every σ-primary ideal is also δ-primary.*

Proof. Let A be a σ-primary ideal of X and let $a, b \in X$ be such that $a \wedge b \in A$ and $a \notin A$. Then $b \in \sigma(A) \subseteq \delta(A)$ by assumption. Hence A is a δ-primary ideal of X. □

Corollary 3.7. *Let σ be an expansion of ideals in a commutative BCK-algebra X. Then every prime ideal of X is σ-primary.*

Proof. Let G be a prime ideal of X. Then G is σ_0-primary, and $\sigma_0(G) = G \subseteq \sigma(G)$. It follows from Theorem 3.6 that G is a σ-primary ideal of X. □

Theorem 3.8. *Let α and β be expansions of subalgebras* (resp., *ideals*) *in X. Let $\sigma : \mathbb{S}(X) \to \mathbb{S}(X)$* (resp., $\sigma : \mathbb{I}(X) \to \mathbb{I}(X)$) *be a function defined by $\sigma(G) = \alpha(G) \cap \beta(G)$ for all $G \in \mathbb{S}(X)$* (resp., $\mathbb{I}(X)$). *Then σ is an expansion of subalgebras* (resp., *ideals*) *in X.*

Proof. For every $G \in \mathbb{S}(X)$ (resp., $\mathbb{I}(X)$), we have $G \subseteq \alpha(G)$ and $G \subseteq \beta(G)$ by (o1), and so $G \subseteq \alpha(G) \cap \beta(G) = \sigma G)$. Let $G, H \in \mathbb{S}(X)$ (resp., $\mathbb{I}(X)$) be such that $G \subseteq H$. Then $\alpha(G) \subseteq \alpha(H)$ and $\beta(G) \subseteq \beta(H)$ by (o2), which imply that

$$\sigma(G) = \alpha(G) \cap \beta(G) \subseteq \alpha(H) \cap \beta(H) = \sigma(H).$$

Therefore σ is an expansion of subalgebras (resp., ideals) in X. □

Generally, the intersection of expansions of subalgebras (resp., ideals) is an expansion of subalgebras (resp., ideals).

Theorem 3.9. *Let X be a commutative BCK-algebra and let σ be an expansion of ideals in X. If $\{J_i \mid i \in D\}$ is a directed collection of σ-primary ideals of X where D is an index set, then the ideal $J := \bigcup_{i \in D} J_i$ is σ-primary.*

Proof. Let $a, b \in X$ be such that $a \wedge b \in J$ and $a \notin J$. Then there exists J_i such that $a \wedge b \in J_i$ and $a \notin J_i$. Since J_i is σ-primary and $J_i \subseteq J$, it follows that $b \in \sigma(J_i) \subseteq \sigma(J)$ so that J is σ-primary. □

Theorem 3.10. *Let σ be an expansion of ideals in a commutative BCK-algebra X. If P is a σ-primary ideal of X, then*

$$(\forall I, J \in \mathbb{I}(X))\,(I \wedge J \subseteq P,\ I \nsubseteq P \ \Rightarrow\ J \subseteq \sigma(P)),$$

where $I \wedge J = \{x \wedge y \mid x \in I,\ y \in J\}$.

Proof. Assume that P is a σ-primary ideal of X and let $I, J \in \mathbb{I}(X)$ be such that $I \wedge J \subseteq P$ and $I \nsubseteq P$. Suppose that $J \nsubseteq \sigma(P)$. Then there exist $a \in I \setminus P$ and $b \in J \setminus \sigma(P)$, which imply that $a \wedge b \in I \wedge J \subseteq P$. But $a \notin P$ and $b \notin \sigma(P)$. This contradicts the assumption that P is σ-primary. Consequently, the result is valid. □

Theorem 3.11. *Let X be a commutative BCK-algebra. If σ is an expansion of ideals in X, then the function $E_\sigma : \mathbb{I}(X) \to \mathbb{I}(X)$ defined by*

$$E_\sigma(G) := \cap\{H \in \mathbb{I}(X) \mid G \subseteq H, \text{ and } H \text{ is } \sigma\text{-primary}\}$$

for all $G \in \mathbb{I}(X)$ is an expansion of ideals in X.

Proof. Clearly, $G \subseteq E_\sigma(G)$ for all $G \in \mathbb{I}(X)$. Let $I, J \in \mathbb{I}(X)$ be such that $I \subseteq J$. Then

$$\begin{aligned} E_\sigma(I) &= \cap\{H \in \mathbb{I}(X) \mid I \subseteq H \text{ and } H \text{ is } \sigma\text{-primary}\} \\ &\subseteq \cap\{H \in \mathbb{I}(X) \mid J \subseteq H \text{ and } H \text{ is } \sigma\text{-primary}\} \\ &= E_\sigma(J). \end{aligned}$$

Hence E_σ is an expansion of ideals in X. □

For any ideals P and Q of a commutative BCK-algebra X, the *residual division* of P and Q is defined to be the ideal

$$P : Q = \bigcap_{x \in Q} x^{-1}P = \{y \in X \mid x \wedge y \in P \text{ for all } x \in Q\}.$$

Theorem 3.12. *Let σ be an expansion of ideals in a commutative BCK-algebra X and let P be a σ-primary ideal of X. Then*

(i) *if I is an ideal of X which is not contained in $\sigma(P)$, then $P : I = P$.*
(ii) *if J is any ideal of X, then $P : J$ is σ-primary.*

Proof. (i) Obviously, $P \subseteq P : I$. Also we have $I \wedge (P : I) \subseteq P$ by the definition of $P : I$. Since $I \nsubseteq \sigma(P)$, it follows from Theorem 3.10 that $P : I \subseteq P$. Therefore $P : I = P$.

(ii) Let $a, b \in X$ be such that $a \wedge b \in P : J$ and $a \notin P : J$. Then $a \wedge x \notin P$ for some $x \in J$. But $(a \wedge x) \wedge b = (a \wedge b) \wedge x \in P$, and so $b \in \sigma(P) \subseteq \sigma(P : J)$. Thus $P : J$ is σ-primary. This completes the proof. □

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