# WEAK AND STRONG CONVERGENCES OF ISHIKAWA ITERATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. Let $C$ be a closed convex subset of a Banach space which satisfies Opial's condition. We first prove that if $T : C \to C$ is asymptotically nonexpansive in the intermediate sense, the Ishikawa iteration process with errors defined by $x_1 \in C$, $x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n$, and $y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n$ converges weakly to some fixed point of $T$, which generalizes the result due to Tan and Xu. Further, we show that if $S$ and $T$ are both comact and asymptotically nonexpansive in the intermediate sense, the iterations $\{x_n\}$ and $\{y_n\}$ defined by $x_1 \in C$, $x_{n+1} = \alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n$, and $y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n$ converge strongly to the same common fixed point of $S$ and $T$, which generalizes the result due to Rhoades.

## 1. Introduction

Let $C$ be a closed convex subset of a Banach space $X$ and let $T$ be a mapping of $C$ into itself. Then $T$ is said to be *asymptotically nonexpansive* [4] if there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \boldsymbol{N}$, where $\boldsymbol{N}$ denotes the set of all positive integers. In particular, if $k_n = 1$ for all $n \in \boldsymbol{N}$, $T$ is said to be *nonexpansive.* The weaker definition (cf. Kirk [7]) requires that

$$\overline{\lim}_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for each $x \in C$, and that $T^N$ is continuous for some $N \in \boldsymbol{N}$. Consider a definition somewhere between these two. $T$ is said to be *asymptotically nonexpansive in the intermediate sense* [1] provided $T$ is uniformly continuous and

$$\overline{\lim}_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

Recall that a Banach space $X$ is said to be *uniformly convex* if the modulus of convexity $\delta_X = \delta_X(\varepsilon)$, $0 < \varepsilon \le 2$, of $X$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \;\Big|\; x, y \in X,\ \|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon\right\}$$

satisfies the inequality $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. A Banach space $X$ is said to satisfy *Opial's condition* [9] if for any sequence $\{x_n\}$ in $X$, $x_n \rightharpoonup x$ implies that

$$\underline{\lim}_{n\to\infty} \|x_n - x\| < \underline{\lim}_{n\to\infty} \|x_n - y\|$$

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for all $y \in X$ with $y \neq x$.

Recently, for a mapping $T$ of $C$ into itself, Tan and Xu [14] considered the following modified Ishikawa iteration process (cf. Ishikawa [5]) in $C$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. They proved that if $X$ is a uniformly convex Banach space which satisfies Opial's condition, $C$ is a bounded closed convex subset of $X$, and $T$ is an asymptotically nonexpansive mapping of $C$ into itself such that $\sum_{n=1}^{\infty}(k_n - 1)$ converges, then for any $x_1$ in $C$, the sequence $\{x_n\}$ defined by (1.1) converges weakly to some fixed point of $T$ under the assumptions that $\{\alpha_n\}$ is bounded away from 0 and 1 and $\{\beta_n\}$ is bounded away from 1. We consider a more general iterative process (cf. Xu [15]) emphasizing the randomness of errors as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0,1]$ satisfying

$$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1 \text{ for all } n \in \boldsymbol{N}, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \gamma_n < \infty \text{ and } \sum_{n=1}^{\infty} \gamma'_n < \infty, \tag{1.4}$$

and $\{u_n\}, \{v_n\}$ are two sequences in $C$. If $\gamma_n = \gamma'_n = 0$ for all $n \in \boldsymbol{N}$, then the iteration process (1.2) reduces to the Ishikawa iteration process [5], while setting $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \in \boldsymbol{N}$, (1.2) reduces to the Mann iteration process with errors, which is a generalized case of the Mann iteration process [8].

In this paper, we first prove a weak convergence theorem of the Ishikawa (and Mann) iteration process with errors defined by (1.2) for a non-Lipschitzian self-mapping, which generalizes the result due to Tan and Xu [14]. Next, let $S, T$ be compact and asymptotically nonexpansive mappings of $C$ into itself in the intermediate sense. Then we shall show a strong convergence theorem for the iterations $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n, \\ y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0,1]$ satisfying (1.3) and (1.4) and $\{u_n\}, \{v_n\}$ are two sequences in $C$, which generalize the result due to Rhoades [11]. Further, we prove a weak convergence theorem for (1.5) without the compactness of $S$ and $T$.

## 2. Weak Convergence Theorems

We first begin with the following:

**Theorem 2.1** ([1]). *Suppose a Banach space $X$ has the uniform $\tau$-Opial property, $C$ is a norm-bounded, sequentially $\tau$-compact subset of $X$, and $T : C \to C$ is asymptotically nonexpansive in the weak sense. If $\{y_n\}$ is a sequence in $C$ such that $\lim_{n\to\infty} \|y_n - z\|$ exists* *for each fixed point* $z$ *of* $T$, *and if* $\{y_n - T^k y_n\}$ *is* $\tau$*-convergent to* $0$ *for each* $k \in \boldsymbol{N}$, *then* $\{y_n\}$ *is* $\tau$*-convergent to a fixed point of* $T$.

**Lemma 2.2** ([14]). *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of nonnegative real numbers such that* $\sum_{n=1}^{\infty} b_n < \infty$ *and*

$$a_{n+1} \leq a_n + b_n$$

*for all* $n \in \boldsymbol{N}$. *Then* $\lim_{n\to\infty} a_n$ *exists.*

**Lemma 2.3** ([12]). *Let* $X$ *be a uniformly convex Banach space, let* $0 < b \leq t_n \leq c < 1$ *for all* $n \in \boldsymbol{N}$, *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences of* $X$ *such that* $\overline{\lim}_{n\to\infty} \|x_n\| \leq a$, $\overline{\lim}_{n\to\infty} \|y_n\| \leq a$ *and* $\overline{\lim}_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$ *for some* $a \geq 0$. *Then, it holds that* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

In this paper, the iterations defined by (1.2) and (1.5) are always assumed that $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are real sequences in $[0, 1]$ satisfying (1.3) and (1.4) and $\{u_n\}$, $\{v_n\}$ are bounded sequences in $C$. Our Theorem 2.11 carries over Theorem 3.2 of Tan and Xu [14] to a more general Ishikawa type process and a non-Lipschitzian self-mapping.

**Lemma 2.4.** *Let* $C$ *be a closed convex subset of a uniformly convex Banach space* $X$ *and let* $S, T$ *be mappings of* $C$ *into itself satisfying that* $F(S) \cap F(T) \neq \emptyset$. *For* $z \in F(S) \cap F(T)$, *put*

$$c_n = \sup_{x\in C}(\|S^n x - z\| - \|x - z\|) \vee \sup_{x\in C}(\|T^n x - z\| - \|x - z\|) \vee 0$$

*for all* $n \in \boldsymbol{N}$. *Suppose that* $\sum_{n=1}^{\infty} c_n < \infty$ *and the sequence* $\{x_n\}$ *is defined by* (1.5). *Then* $\lim_{n\to\infty} \|x_n - z\|$ *exists.*

*Proof.* Since $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M = \sup_{n\in\boldsymbol{N}} \|u_n - z\| \vee \sup_{n\in\boldsymbol{N}} \|v_n - z\| \ (< \infty).$$

Since

$$\begin{aligned} \|S^n y_n - z\| &\leq \|y_n - z\| + c_n \\ &= \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - z\| + c_n \\ &\leq \alpha'_n \|x_n - z\| + \beta'_n \|T^n x_n - z\| + \gamma'_n \|v_n - z\| + c_n \\ &\leq \alpha'_n \|x_n - z\| + \beta'_n\{\|x_n - z\| + c_n\} + \gamma'_n \|v_n - z\| + c_n \\ &\leq (1-\gamma'_n) \|x_n - z\| + \gamma'_n \|v_n - z\| + 2c_n, \end{aligned} \tag{2.1}$$

we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \|\alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|S^n y_n - z\| + \gamma_n \|u_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n\{(1-\gamma'_n) \|x_n - z\| + \gamma'_n \|v_n - z\| + 2c_n\} \\ &\quad + \gamma_n \|u_n - z\| \\ &\leq (1 - (\gamma_n + \beta_n\gamma'_n)) \|x_n - z\| + \gamma'_n M + 2c_n + \gamma_n M \\ &\leq \|x_n - z\| + (\gamma'_n + \gamma_n)M + 2c_n. \end{aligned}$$

By Lemma 2.2, we readily see that $\lim_{n\to\infty} \|x_n - z\|$ exists. □

**Lemma 2.5.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $X$ and let $T$ be a mapping of $C$ into itself such that $F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. *$a \leq \alpha_n, \beta_n \leq b,\ 0 \leq \beta'_n \leq b$ for some $a, b \in (0,1)$, or*
2. *$a \leq \beta_n \leq 1,\ a \leq \alpha'_n, \beta'_n \leq b$ for some $a, b \in (0,1)$.*

*Then both $\{T^n x_n - x_n\}$ and $\{x_n - y_n\}$ converge strongly to $0$.*

*Proof.* Take $z \in F(T)$ and let $r = \lim_{n\to\infty} \|x_n - z\|$ which exists by Lemma 2.4. Note that $d_n \equiv \max\{\gamma'_n, \gamma_n/a\} \to 0$ as $n \to \infty$. Since $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M = \sup_{n\in\boldsymbol{N}} \|u_n - z\| \vee \sup_{n\in\boldsymbol{N}} \|v_n - z\| \ (< \infty).$$

Now, we assume (1). Since $\|T^n y_n - z\| \leq \|x_n - z\| + d_n M + 2c_n$ by the same calculation as (2.1) and $\left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq \|x_n - z\| + d_n M$, we get $\overline{\lim}_{n\to\infty} \|T^n y_n - z\| \leq r$ and

$$\overline{\lim}_{n\to\infty} \left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \leq r.$$

On the other hand,

$$\begin{aligned} r &= \lim_{n\to\infty} \|x_{n+1} - z\| \\ &= \lim_{n\to\infty} \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\ &= \lim_{n\to\infty} \left\| \beta_n (T^n y_n - z) + (1 - \beta_n) \left( \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right) \right\|. \end{aligned}$$

By Lemma 2.3, it holds that $\lim_{n\to\infty} \left\| T^n y_n - \frac{\alpha_n x_n}{\alpha_n + \gamma_n} - \frac{\gamma_n u_n}{\alpha_n + \gamma_n} \right\| = 0$, and so we obtain $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$ by virtue of $\sup_{n\in\boldsymbol{N}} \|x_n - u_n\| < \infty$. Since

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\leq \|x_n - y_n\| + c_n + \|T^n y_n - x_n\| \\ &= \|x_n - \alpha'_n x_n - \beta'_n T^n x_n - \gamma'_n v_n\| + \|T^n y_n - x_n\| + c_n \\ &\leq \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\| + c_n, \end{aligned}$$

we have

$$\begin{aligned} (1 - b) \|T^n x_n - x_n\| &\leq (1 - \beta'_n) \|T^n x_n - x_n\| \\ &\leq \gamma'_n \|x_n - v_n\| + \|T^n y_n - x_n\| + c_n \\ &\leq \gamma'_n M' + \|T^n y_n - x_n\| + c_n, \end{aligned} \tag{2.2}$$

where $M' = \sup_{n\in\boldsymbol{N}} \|x_n - v_n\| \ (< \infty)$. We easily have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0 \tag{2.3}$$

from (2.2). Next, assuming (2), we have

$$\|x_{n+1} - z\| = \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\|$$

$$\begin{aligned}
&\le \alpha_n \|x_n - z\| + \beta_n \|T^n y_n - z\| + \gamma_n \|u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \{\|y_n - z\| + c_n\} + \gamma_n M \\
&\le (1-\beta_n) \|x_n - z\| + \beta_n \|y_n - z\| + c_n + \gamma_n M
\end{aligned}$$

and hence

$$\frac{\|x_{n+1} - z\| - \|x_n - z\|}{\beta_n} + \|x_n - z\| \le \|y_n - z\| + \frac{c_n}{a} + \frac{\gamma_n}{a} M.$$

So, using $\|y_n - z\| \le \|x_n - z\| + c_n + d_n M$ obtained by (2.1), we have

$$r \le \underline{\lim}_{n\to\infty} \|y_n - z\| \le \overline{\lim}_{n\to\infty} \|y_n - z\| \le \overline{\lim}_{n\to\infty} \{\|x_n - z\| + c_n + d_n M\} = r.$$

Hence

$$\begin{aligned}
r &= \lim_{n\to\infty} \|y_n - z\| \\
&= \lim_{n\to\infty} \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - z\| \\
&= \lim_{n\to\infty} \left\| \beta'_n (T^n x_n - z) + (1-\beta'_n) \left( \frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right) \right\|.
\end{aligned}$$

Further, it holds that $\overline{\lim}_{n\to\infty} \|T^n x_n - z\| \le r$ from $\|T^n x_n - z\| \le \|x_n - z\| + c_n$ and

$$\overline{\lim}_{n\to\infty} \left\| \frac{\alpha'_n x_n}{\alpha'_n + \gamma'_n} + \frac{\gamma'_n v_n}{\alpha'_n + \gamma'_n} - z \right\| \le r$$

similarly to the arguement above. So, using Lemma 2.3 and $\sup_{n\in \boldsymbol{N}} \|x_n - v_n\| < \infty$, we also have (2.3). Finally, we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$ immediately by (1.2) and (2.3). □

**Lemma 2.6.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $X$ and let $T$ be a mapping of $C$ into itself satisfying that $F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. *$a \le \alpha_n, \beta_n \le b$, $0 \le \beta'_n \le b$ for some $a, b \in (0,1)$, or*
2. *$a \le \beta_n \le 1$, $a \le \alpha'_n, \beta'_n \le b$ for some $a, b \in (0,1)$.*

*Then $\{x_n - Tx_n\}$ converges strongly to $0$.*

*Proof.* Since

$$\begin{aligned}
\|x_n - x_{n+1}\| &\le \|x_n - T^n x_n\| + \|T^n x_n - x_{n+1}\| \\
&\le (\alpha_n + 1) \|x_n - T^n x_n\| + \beta_n \|T^n x_n - T^n y_n\| + \gamma_n \|T^n x_n - u_n\| \\
&\le (\alpha_n + 1) \|x_n - T^n x_n\| + \beta_n (\|x_n - y_n\| + c_n) + \gamma_n \|u_n - T^n x_n\|,
\end{aligned}$$

we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ by Lemma 2.5. Further, since

$$\begin{aligned}
\|x_n - Tx_n\| \le\ & \|x_n - x_{x+1}\| + \left\|x_{n+1} - T^{n+1} x_{n+1}\right\| \\
&+ \left\|T^{n+1} x_{n+1} - T^{n+1} x_n\right\| + \left\|T^{n+1} x_n - T x_n\right\| \\
\le\ & 2\|x_n - x_{x+1}\| + \left\|x_{n+1} - T^{n+1} x_{n+1}\right\| + c_{n+1} + \left\|T^{n+1} x_n - T x_n\right\|,
\end{aligned}$$

we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ by Lemma 2.5 and the uniform continuity of $T$. □

**Theorem 2.7.** *Let $C$ be a bounded closed convex subset of a uniformly convex Banach space $X$ which satisfies Opial's condition and let $T$ be an asymptotically nonexpansive mapping of $C$ into itself in the intermediate sense. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. *$a \le \alpha_n, \beta_n \le b,\ 0 \le \beta'_n \le b$ for some $a, b \in (0,1)$, or*
2. *$a \le \beta_n \le 1,\ a \le \alpha'_n, \beta'_n \le b$ for some $a, b \in (0,1)$.*

*Then $\{x_n\}$ converges weakly to some fixed point of $T$. Further, the two limits of $\{x_n\}$ and $\{y_n\}$ coincide.*

*Proof.* The existence of a fixed point of $T$ follows form Kirk [7]. By Lemma 2.6 we have

$$\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$$

for all $m \in \boldsymbol{N}$. Now, we can apply Theorem 2.1 with the weak topology instead of $\tau$-topology and get the conclusion. Further, the two limits of $\{x_n\}$ and $\{y_n\}$ coincide by Lemma 2.5. □

As a direct consequence, taking $\beta'_n = \gamma'_n = 0$ for $n \in \boldsymbol{N}$ in Theorem 2.7, we have the following result, which carries over a more general Mann type process and a non-Lipschitzian self-mapping.

**Theorem 2.8.** *Let $X$ be a uniformly convex Banach space which satisfies Opial's condition and let $C$ be a bounded closed convex subset of $X$. Let $T$ be an asymptotically nonexpansive mapping of $C$ into itself in the intermediate sense. Put*

$$c_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and a sequence $\{x_n\}$ is defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n u_n,$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ satisfying $a \le \alpha_n, \beta_n \le b$ for some $a, b \in (0,1)$, $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \boldsymbol{N}$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is a sequence in $C$. Then $\{x_n\}$ converges weakly to some fixed point of $T$.*

Next, we consider the weak convergence of the sequence $\{x_n\}$ defined by (1.5).

**Lemma 2.9.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $X$. Let $S, T$ be asymptotically nonexpansive mappings of $C$ into itself in the intermediate sense with $F(S) \cap F(T) \ne \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by (1.5) satisfies $a \le \alpha_n, \alpha'_n, \beta_n, \beta'_n \le b$ for some $a, b \in (0,1)$. Then, we have $\lim_{n\to\infty} \|S^n x_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. Further, it holds that $\lim_{n\to\infty} \|x_n - S x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T x_n\| = 0$, and $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

*Proof.* Take $z \in F(S) \cap F(T)$ and put $r = \lim_{n\to\infty} \|x_n - z\|$ which exists by Lemma 2.4. Note that $d_n \equiv \max\{\gamma_n', \gamma_n/a\} \to 0$ as $n \to \infty$. Since $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M = \sup_{n\in\boldsymbol{N}} \|u_n - z\| \vee \sup_{n\in\boldsymbol{N}} \|v_n - z\| \ \ (< \infty).$$

Since $\|S^n y_n - z\| \le \|x_n - z\| + d_n M + 2c_n$ by (2.1) and $\left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \le \|x_n - z\| + d_n M$, we get $\overline{\lim}_{n\to\infty} \|S^n y_n - z\| \le r$ and $\overline{\lim}_{n\to\infty} \left\| \frac{\alpha_n x_n}{\alpha_n + \gamma_n} + \frac{\gamma_n u_n}{\alpha_n + \gamma_n} - z \right\| \le r$, and so we obtain

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0 \tag{2.4}$$

as in the proof of Lemma 2.5. Since

$$\begin{aligned} \|x_n - z\| &\le \|x_n - S^n y_n\| + \|S^n y_n - z\| \\ &\le \|x_n - S^n y_n\| + \|y_n - z\| + c_n \end{aligned}$$

and

$$\begin{aligned} \|y_n - z\| &\le \|\alpha_n' x_n + \beta_n' T^n x_n + \gamma_n' v_n - z\| \\ &\le \alpha_n' \|x_n - z\| + \beta_n' \|T^n x_n - z\| + \gamma_n' \|v_n - z\| \\ &\le \alpha_n' \|x_n - z\| + \beta_n' \{\|x_n - z\| + c_n\} + \gamma_n' \|v_n - z\| \\ &\le (1 - \gamma_n') \|x_n - z\| + c_n + \gamma_n' M \\ &\le \|x_n - z\| + c_n + \gamma_n' M, \end{aligned}$$

we have

$$\begin{aligned} r &\le \underline{\lim}_{n\to\infty} \{\|x_n - S^n y_n\| + \|y_n - z\| + c_n\} \\ &= \underline{\lim}_{n\to\infty} \|y_n - z\| \\ &\le \overline{\lim}_{n\to\infty} \|y_n - z\| \\ &\le \overline{\lim}_{n\to\infty} \{\|x_n - z\| + c_n + \gamma_n' M\} = r \end{aligned}$$

and thus

$$\begin{aligned} r &= \lim_{n\to\infty} \|y_n - z\| \\ &= \lim_{n\to\infty} \|\alpha_n' x_n + \beta_n' T^n y_n + \gamma_n' v_n - z\| \\ &= \lim_{n\to\infty} \left\| \beta_n' (T^n x_n - z) + (1 - \beta_n') \left( \frac{\alpha_n' x_n}{\alpha_n' + \gamma_n'} + \frac{\gamma_n' v_n}{\alpha_n' + \gamma_n'} - z \right) \right\|. \end{aligned}$$

It is easily that $\overline{\lim}_{n\to\infty} \|T^n x_n - z\| \le r$ and $\overline{\lim}_{n\to\infty} \left\| \frac{\alpha_n' x_n}{\alpha_n' + \gamma_n'} + \frac{\gamma_n' v_n}{\alpha_n' + \gamma_n'} - z \right\| \le r$. So, using Lemma 2.3 and $\sup_{n\in\boldsymbol{N}} \|x_n - v_n\| < \infty$, we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{2.5}$$

On the other hand, from $\|x_{n+1} - x_n\| \le \beta_n \|S^n y_n - x_n\| + \gamma_n \|u_n - x_n\|$, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

by (2.4). Since

$$\begin{aligned} \|x_n - Tx_n\| \le \|x_n - x_{x+1}\| &+ \left\|x_{n+1} - T^{n+1} x_{n+1}\right\| \\ &+ \left\|T^{n+1} x_{n+1} - T^{n+1} x_n\right\| + \left\|T^{n+1} x_n - T x_n\right\| \end{aligned}$$

$$\leq 2 \|x_n - x_{x+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + c_n + \left\|T^{n+1}x_n - Tx_n\right\|,$$

we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ by $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, the uniform continuity of $T$ and (2.5). Set $M' = \sup_{n\in \boldsymbol{N}} \|x_n - v_n\|$. Since

$$\begin{aligned}
\|S^n x_n - x_n\| &\leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\
&\leq \|x_n - y_n\| + c_n + \|S^n y_n - x_n\| \\
&= \|x_n - \alpha'_n x_n - \beta'_n T^n x_n - \gamma'_n v_n\| + c_n + \|S^n y_n - x_n\| \\
&\leq \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|x_n - v_n\| + c_n + \|S^n y_n - x_n\| \\
&\leq b\|T^n x_n - x_n\| + \gamma'_n M' + c_n + \|S^n y_n - x_n\|,
\end{aligned}$$

we obtain $\lim_{n\to\infty} \|S^n x_n - x_n\| = 0$ by (2.4) and (2.5). Therefore, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0$$

similarly to the arguement above. Finally, since $\|x_n - y_n\| \leq b\|T^n x_n - x_n\| + \gamma'_n M'$, we obtain $\lim_{n\to\infty} \|x_n - y_n\| = 0$. □

**Theorem 2.10.** *Let $C$ be a bounded closed convex subset of a uniformly convex Banach space $X$ satisfying Opial's condition. Let $S, T$ be asymptotically nonexpansive mappings of $C$ into itself in the intermediate sense with $F(S) \cap F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by (1.5) satisfies $a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $S$ and $T$. Further, the two limits of $\{x_n\}$ and $\{y_n\}$ coincide.*

*Proof.* We have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ from Lemma 2.9 and so $\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$ for all $m \in \boldsymbol{N}$ by the uniform continuity of $T$. Hence, by Theorem 2.1 there exists $z_1 \in F(T)$ such that $x_n \rightharpoonup z_1$. Similarly, there exists $z_2 \in F(S)$ such that $x_n \rightharpoonup z_2$. Hence, $z_1 = z_2 \in F(S) \cap F(T)$ by the uniqueness of limits. Further, the two limits of $\{x_n\}$ and $\{y_n\}$ coincide by Lemma 2.9. □

As a direct consequence of Theorem 2.7 and Theorem 2.10 we improve Theorem 3.2 due to Tan and Xu [14] to a more general Ishikawa type process (1.2) instead of (1.1).

**Theorem 2.11.** *Let $X$ be a uniformly convex Banach space which satisfies Opial's condition and let $C$ be a bounded closed convex subset of $X$. Let $T$ be an asymptotically nonexpansive self-mapping of $C$ such that $\sum_{n=1}^{\infty} (k_n - 1)$ converges. Suppose that the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. *$a \leq \alpha_n, \beta_n \leq b,\ 0 \leq \beta'_n \leq b$ for some $a, b \in (0, 1)$,*
2. *$a \leq \beta_n \leq 1,\ a \leq \alpha'_n, \beta'_n \leq b$ for some $a, b \in (0, 1)$, or*
3. *$a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b$ for some $a, b \in (0, 1)$.*

*Then $\{x_n\}$ converges weakly to some fixed point of $T$.*

*Proof.* We may assume that $k_n \geq 1$ for all $n \in \boldsymbol{N}$. Note that

$$\sum_{n=1}^{\infty} c_n \leq \sum_{n=1}^{\infty} (k_n - 1) \sup_{x,y\in C} \|x - y\| < \infty.$$

The conclusion now follows easily from Theorem 2.7 and Theorem 2.10. □

## 3. Strong Convergence Theorems

The following Theorem 3.4 and Theorem 3.6 carry over Theorem 3 due to Rhoades [11] to a more general Ishikawa type process and a non-Lipschitzian self-mapping.

**Theorem 3.1.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ and let $T$ be an asymptotically nonexpansive mapping of $C$ into itself in the intermediate sense with a fixed point. Put*

$$c_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$, the sequence $\{x_n\}$ defined by* (1.2) *satisfies either*

1. $a \le \alpha_n, \beta_n \le b,\ 0 \le \beta'_n \le b$ *for some $a, b \in (0,1)$, or*
2. $a \le \beta_n \le 1,\ a \le \alpha'_n, \beta'_n \le b$ *for some $a, b \in (0,1)$*

*and $T(C) \cup \{u_n\}$ is contained in a compact subset of $C$. Then $\{x_n\}$ converges strongly to some fixed point of $T$.*

*Proof.* By Mazur's theorem [3], $\overline{\mathrm{co}}(\{x_1\} \cup T(C) \cup \{u_n\})$ is a compact subset of $C$ containing $\{x_n\}$. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $z \in C$ such that $x_{n_k} \to z$. By the boundedness of $\{u_n\}$, $\lim_{n\to\infty} \gamma_n = 0$ and Lemma 2.5, we have

$$\begin{aligned}
\|x_{n_k+1} - x_{n_k}\| &\le \beta_{n_k}(\|T^{n_k} y_{n_k} - T^{n_k} x_{n_k}\| + \|T^{n_k} x_{n_k} - x_{n_k}\|) \\
&\quad + \gamma_{n_k} \|u_{n_k} - x_{n_k}\| \\
&\le \beta_{n_k}(\|x_{n_k} - y_{n_k}\| + c_{n_k} + \|T^{n_k} x_{n_k} - x_{n_k}\|) \\
&\quad + \gamma_{n_k} \|u_{n_k} - x_{n_k}\| \\
&\to 0 \quad (k \to \infty).
\end{aligned}$$

Therefore, from the uniform continuity of $T$ and Lemma 2.5, we obtain

$$\begin{aligned}
\|z - Tz\| &\le \|z - x_{n_k+1}\| + \left\|x_{n_k+1} - T^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\right\| + \left\|T^{n_k+1} x_{n_k} - Tz\right\| \\
&\le \|z - x_{n_k+1}\| + \left\|x_{n_k+1} - T^{n_k+1} x_{n_k+1}\right\| + \|x_{n_k+1} - x_{n_k}\| \\
&\quad + c_{n_k+1} + \left\|T^{n_k+1} x_{n_k} - Tz\right\| \\
&\to 0 \quad (k \to \infty),
\end{aligned}$$

which implies that $z$ is a fixed point of $T$. By Lemma 2.4 $\lim_{n\to\infty} \|x_n - z\|$ exists, and so we have $\lim_{n\to\infty} \|x_n - z\| = 0$. □

**Theorem 3.2.** *Let $C$ be a compact convex subset of a uniformly convex Banach space $E$ and let $T$ be an asymptotically nonexpansive mapping of $C$ into itself in the intermediate sense. Put*

$$c_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by* (1.2) *satisfies either*

1. $a \le \alpha_n, \beta_n \le b,\ 0 \le \beta'_n \le b$ *for some $a, b \in (0,1)$, or*
2. $a \le \beta_n \le 1,\ a \le \alpha'_n, \beta'_n \le b$ *for some $a, b \in (0,1)$.*

*Then $\{x_n\}$ converges strongly to some fixed point of $T$.*

*Proof.* The existence of a fixed point follows from Schauder's fixed point theorem. So, we have the desired result by Theorem 3.1 immediately. □

As a direct consequence of Theorem 3.2, we have the following result.

**Corollary 3.3.** *Let $C$ be a compact convex subset of a uniformly convex Banach space $E$ and let $T$ be an asymptotically nonexpansive mapping of $C$ into itself with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose that the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. $a \leq \alpha_n, \beta_n \leq b,\ 0 \leq \beta'_n \leq b$ *for some* $a, b \in (0,1)$, *or*
2. $a \leq \beta_n \leq 1,\ a \leq \alpha'_n, \beta'_n \leq b$ *for some* $a, b \in (0,1)$.

*Then $\{x_n\}$ converges strongly to some fixed point of $T$.*

**Theorem 3.4.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ and let $T$ be a compact and asymptotically nonexpansive mapping of $C$ into itself in the intermediate sense with a fixed point. Put*

$$c_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$, the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. $a \leq \alpha_n, \beta_n \leq b,\ 0 \leq \beta'_n \leq b$ *for some* $a, b \in (0,1)$, *or*
2. $a \leq \beta_n \leq 1,\ a \leq \alpha'_n, \beta'_n \leq b$ *for some* $a, b \in (0,1)$.

*Then $\{x_n\}$ converges strongly to some fixed point of $T$.*

*Proof.* $\{x_n\}$ is bounded by Lemma 2.4 and $T$ is compact, so that there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $z \in C$ such that $Tx_{n_i} \to z$. It is easily follows from the continuity of $T$ and Lemma 2.6 that $z$ is a fixed point of $T$ and $x_{n_i} \to z$. Therefore, $\{x_n\}$ converges strongly to $z$ by Lemma 2.4. □

Next, we consider the strong convergence of the sequence $\{x_n\}$ defined by (1.5).

**Theorem 3.5.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $E$ and let $S, T$ be an asymptotically nonexpansive mapping of $C$ into itself in the intermediate sense with $F(S) \cap F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $\sum_{n=1}^{\infty} c_n < \infty$, the sequence $\{x_n\}$ defined by (1.5) satisfies $a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b$ for some $a, b \in (0,1)$, and $S(C) \cup \{u_n\}$ is contained in a compact subset of $C$. Then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.*

*Proof.* By Mazur's theorem [3], $\overline{\text{co}}(\{x_1\} \cup S(C) \cup \{u_n\})$ is a compact subset of $C$ containing $\{x_n\}$. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $z \in C$ such that $x_{n_k} \to z$. By the boundedness of $\{u_n\}$, $\lim_{n\to\infty} \gamma_n = 0$ and Lemma 2.9, we have

$$\begin{aligned}
\|x_{n_k+1} - x_{n_k}\| &\leq \beta_{n_k}(\|S^{n_k} y_{n_k} - S^{n_k} x_{n_k}\| + \|S^{n_k} x_{n_k} - x_{n_k}\|) \\
&\quad + \gamma_{n_k}\|u_{n_k} - x_{n_k}\| \\
&\leq \beta_{n_k}(\|x_{n_k} - y_{n_k}\| + c_{n_k} + \|S^{n_k} x_{n_k} - x_{n_k}\|) \\
&\quad + \gamma_{n_k}\|u_{n_k} - x_{n_k}\| \\
&\to 0 \quad (k \to \infty).
\end{aligned}$$

Therefore, from the uniform continuity of $S$ and Lemma 2.9, we obtain

$$
\begin{aligned}
\|z - Sz\| &\leq \|z - x_{n_k+1}\| + \left\|x_{n_k+1} - S^{n_k+1}x_{n_k+1}\right\| \\
&\quad + \left\|S^{n_k+1}x_{n_k+1} - S^{n_k+1}x_{n_k}\right\| + \left\|S^{n_k+1}x_{n_k} - Sz\right\| \\
&\leq \|z - x_{n_k+1}\| + \left\|x_{n_k+1} - S^{n_k+1}x_{n_k+1}\right\| + \|x_{n_k+1} - x_{n_k}\| \\
&\quad + c_{n_k+1} + \left\|S^{n_k+1}x_{n_k} - Sz\right\| \\
&\to 0 \quad (k \to \infty),
\end{aligned} \tag{3.1}
$$

which implies that $z$ is a fixed point of $S$. Further, $z$ is a fixed point of $T$ by the same argument of (3.1). By Lemma 2.4, $\lim_{n\to\infty} \|x_n - z\|$ exists, and so we have $\lim_{n\to\infty} \|x_n - z\| = 0$. □

**Theorem 3.6.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $X$. Let $S, T$ be asymptotically nonexpansive mappings of $C$ into itself in the intermediate sense with $F(S) \cap F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

*for all $n \in \boldsymbol{N}$. Suppose that $S$ is compact, $\sum_{n=1}^{\infty} c_n < \infty$ and the sequence $\{x_n\}$ defined by (1.5) satisfies $a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.*

*Proof.* $\{x_n\}$ is bounded by Lemma 2.4 and $S$ is compact, so that there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $z \in C$ such that $Sx_{n_i} \to z$. Therefore, we have the conclusion by the same argument of the proof of Theorem 3.5. □

As a direct consequence of Theorem 3.4 and Theorem 3.6, we improve Theorem 3 due to Rhoades [11] to a more general Ishikawa type process (1.2) instead of (1.1).

**Corollary 3.7.** *Let $C$ be a closed convex subset of a uniformly convex Banach space $X$. Let $T$ be a completely continuous and asymptotically nonexpansive mapping of $C$ into itself with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that the sequence $\{x_n\}$ defined by (1.2) satisfies either*

1. *$a \leq \alpha_n, \beta_n \leq b$, $0 \leq \beta'_n \leq b$ for some $a, b \in (0, 1)$,*
2. *$a \leq \beta_n \leq 1$, $a \leq \alpha'_n, \beta'_n \leq b$ for some $a, b \in (0, 1)$, or*
3. *$a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b$ for some $a, b \in (0, 1)$.*

*Then $\{x_n\}$ converges strongly to some fixed point of $T$.*

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