LUSIN'S THEOREM AND BOCHNER INTEGRATION

Peter A. Loeb and Erik Talvila *

Received November 19, 2003

Abstract. It is shown that the approximating functions used to define the Bochner integral can be formed using geometrically nice sets, such as balls, from a differentiation basis. Moreover, every appropriate sum of this form will be within a preassigned ε of the integral, with the sum for the local errors also less than ε. All of this follows from the ubiquity of Lebesgue points, which is a consequence of Lusin's theorem, for which a simple proof is included in the discussion.

1 Introduction An attractive feature of the Henstock–Kurzweil integral for the real line is that it can be defined in terms of Riemann sums using intervals. In [11], we showed that for more general spaces, Lebesgue points and points of approximate continuity can be used to approximate a Lebesgue integral $\int_X f \, d\mu$ with sums employing disjoint, geometrically nice sets S_i covering all but a set of measure 0 of X. We also showed that any such sum will be within a preassigned ε of the integral, provided each term $f(x_i)\mu(S_i)$ of the sum has the property that S_i is contained in a ball about x_i of radius at most $\delta(x_i)$, where δ is a "gauge" function determined by f and ε.

In this article, we show that a similar result holds for the Bochner integral, where the domain of the integrand is a measure space $(X, \mathcal{M}, \mu)$ on which is defined a differentiation basis. Although somewhat more general settings are possible, we will assume that X is a finite dimensional normed vector space and the differentiation basis is obtained using the Besicovitch or Morse covering theorem (see Section 3). The geometrically nice sets we will use for the approximating functions will be balls or the more general starlike sets described in Section 3. We will assume that μ is a complete **Radon measure**; i.e., μ is a regular measure on $\mathcal{M}$, which includes the Borel sets, and compact sets have finite measure. By a **measurable** set, we will always mean a set in $\mathcal{M}$. We will let $\mathbb{N}$ denote the natural numbers and $\mathbb{R}$ the real numbers.

In what follows, an integrand f will take its values in a Banach space $(Y, \|\cdot\|)$ and will be **μ-measurable**, meaning, there is a sequence of simple functions f_n with $\lim_n \|f_n - f\| = 0$ μ-a.e. on X. This will allow us to use Lusin's theorem. An elementary proof of that theorem, simple even for real-valued functions on $\mathbb{R}$, is given in the next section.

Our approximating sums will be constructed using a gauge function δ mapping X into $(0, 1]$. In the theory of Henstock–Kurzweil and McShane integration, the appearance of a gauge function is somewhat mysterious. We show in proving Theorem 8 how the properties of Lebesgue points (discussed below in Section 3) can be used to determine an appropriate gauge.

We note that the Bochner integral has been studied in terms of Riemann sums over finite partitions of a metric space in [14] and over generalized McShane partitions of a measure

---

2000 *Mathematics Subject Classification.* Primary 28A20, 28B05; Secondary 26A39.

Key words and phrases. Bochner integral, Lebesgue point, Lusin's theorem, Riemann sum, gauge function.

*The work of both authors here was supported in part by the NSERC and the University of Alberta. The authors thank David Ross for a helpful pointer to the literature in [7] and [10].

space in [8] and [3]. See also [4]. None of these papers, however, works with approximations using geometrically nice sets.

2 Lusin's Theorem Recall that a function f from X into a topological space $(Y, \mathcal{T})$ is **Borel measurable** if the inverse image of each open set in Y is in $\mathcal{M}$. In our case, where Y is a Banach space and f is μ-measurable, it follows from Theorem III.6.10 of [6] that f is Borel measurable on X. We note that it follows from the same theorem and a deep result of D. H. Fremlin (Theorem 2B in [7] or Theorem 4.1 in the expository article [10] by J. Kupka and K. Prikry) that if Y is a metric space and $f : X \to Y$ is Borel measurable, then f is μ-measurable.

In any case, Lusin's theorem holds for the restriction of f to a set $\Omega \subseteq X$ with $\mu(\Omega) < +\infty$, and is easily extended to all of X using the σ-finiteness of μ. Here is the general statement of the theorem for the case of a finite measure space, with an elementary proof that is appropriate even for the simplest setting.

Theorem 1 (Lusin) *Let Y be a topological space with a countable base $\langle V_n \rangle$ for the topology, and let f be a Borel measurable function from a Radon measure space of finite measure $(\Omega, \mathcal{A}, \mu)$ into Y. Given $\varepsilon > 0$, there is a compact set K with $\mu(\Omega \setminus K) < \varepsilon$ such that f restricted to K is continuous.*

Proof: Fix compact sets $K_n \subseteq f^{-1}[V_n]$ and $K'_n \subseteq \Omega \setminus f^{-1}[V_n]$ for each n so that $\mu(\Omega \setminus K) < \varepsilon$ when $K := \bigcap_n (K_n \cup K'_n)$. Given $x \in K$ and an n with $f(x) \in V_n$, $x \in O := \Omega \setminus K'_n$ and $f[O \cap K] \subseteq V_n$. $\square$

Remark 2 In Oxtoby's text [13] a similar principle is used in a more complex proof to show that a measurable $f : \mathbb{R} \to \mathbb{R}$ is continuous when restricted to a large measurable subset of $\mathbb{R}$.

3 Covering Theorems Our integration result is based on calculations using a covering theorem. Here we use either Besicovitch's theorem [1] (also see [9]) for a covering by balls, or the theorem of Morse [12] involving more general sets. In [11], we have established strengthened versions of these theorems that hold for our finite dimensional normed vector space X. Moreover, these covering theorems are also valid for a space that is locally isometric to X since one only needs to bound the cardinality of a finite collection of sets with small diameter forming a "τ-satellite configuration"as defined in [9] and [11]. We leave it to the interested reader to consider our results in this more general setting, as well as settings using Vitali's covering theorem.

We will denote the closed (and compact) ball in X with center a and radius $r > 0$ by $B(a, r) := \{x \in X : \|x - a\| \le r\}$. The sets used by Morse involve a parameter $\lambda \ge 1$; they are closed balls when $\lambda = 1$.

Given $\lambda \ge 1$ and $a \in X$, we say that a set $S(a) \subseteq X$ is a **Morse set** or **λ-Morse set** associated with a and λ if it satisfies two conditions. First, $S(a)$ must be **λ-regular**. This means that there is an $r > 0$ such that $B(a, r) \subseteq S(a) \subseteq B(a, \lambda r)$. Second, $S(a)$ must be **starlike** with respect to $B(a, r)$. This means that for each $y \in B(a, r)$ and each $x \in S(a)$, the line segment $\alpha y + (1 - \alpha)x$, $0 \le \alpha \le 1$, is contained in $S(a)$. We call a the **tag** for $S(a)$. Note that the closure $\mathrm{cl}\,(S(a))$ of a λ-Morse set $S(a)$ is again a λ-Morse set. We say that a λ-Morse set $S(a)$ is **δ-fine** with respect to a gauge function δ defined at a if $S(a) \subseteq B(a, \delta(a))$.

Suppose we are given $\lambda \ge 1$, a Radon measure μ on X, and an open subset Ω of X. We will call a collection $\mathcal{S}$ of λ-Morse sets a **fine**, **measurable**, **λ-Morse cover** of Ω provided each set in $\mathcal{S}$ is a measurable subset of Ω and each $a \in \Omega$ is the tag of sets in $\mathcal{S}$ with arbitrarily small diameters. A sequence $\langle S_i \rangle$ from such an $\mathcal{S}$ is said to be **μ-exhausting of** Ω if it is a finite or countably infinite sequence that is pairwise disjoint, and covers all but a set of μ-measure 0 of Ω. A collection $\mathcal{S}$ is called a **μ-a.e. λ-Morse cover** of Ω if it is a fine, measurable, λ-Morse cover of Ω and for each nonempty open subset $U \subseteq \Omega$ the collection $\mathcal{S}_U = \{S \in \mathcal{S} : S \subseteq U\}$ has the property that for any gauge function $\delta : U \to (0, 1]$, there is a sequence of δ-fine sets in $\mathcal{S}_U$ that is μ-exhausting of U. In [11] we have established the following result for μ and Ω.

Proposition 3 *A fine, measurable, λ-Morse cover $\mathcal{S}$ of Ω is a μ-a.e. λ-Morse cover of Ω if it consists of closed sets or if for each set $S \in \mathcal{S}$, $\mu(\Omega \cap (\mathrm{cl}(S) \setminus S)) = 0$.*

We have also shown that the conditions in Proposition 3 are fulfilled by any measurable Morse cover $\mathcal{S}$ that is **scaled**. This means that for each $S(a) \in \mathcal{S}$ and each $p \in (0, 1]$, the set $S^{(p)}(a)$ is also in $\mathcal{S}$ where $S^{(p)}(a) = \{a + px : a + x \in S(a)\}$.

4 Approximate Continuity and Lebesgue Points For this section, we fix a μ-a.e. λ-Morse cover $\mathcal{S}$ of an open set $\Omega \subseteq X$. We work with a μ-measurable $f : \Omega \to Y$. The following notions depend on the choice of $\mathcal{S}$.

Definition 4 *A point $a \in \Omega$ is a **point of approximate continuity** for f if for all positive ε and η there is an $R > 0$ such that if $S(a)$ is a set in $\mathcal{S}$ with tag a and $S(a) \subseteq B(a, R)$, then for $E(a, \eta) := \{x \in S(a) : \|f(a) - f(x)\| > \eta\}$ we have $\mu(E(a, \eta)) \leq \varepsilon\, \mu(S(a))$. A point $a \in \Omega$ is a **Lebesgue point** of f with respect to $f(a)$ if for any $\varepsilon > 0$ there is an $R > 0$ such that if $S(a)$ is a set in $\mathcal{S}$ with tag a and $S(a) \subseteq B(a, R)$, then*

$$\int_{S(a)} \|f(x) - f(a)\|\, \mu(dx) \leq \varepsilon\, \mu(S(a)).$$

It is easy to see that if $a \in \Omega$ is a Lebesgue point of f with respect to $f(a)$, then a is a point of approximate continuity for f. It follows from the fact that $\mathcal{S}$ is a differentiation basis that if g is a μ-integrable, nonnegative, real-valued function on Ω, then μ-almost all points of Ω are Lebesgue points. (See, for example, [2].) Moreover, if A is a measurable subset of Ω, then almost all points of A are **points of density**, that is, points of approximate continuity with respect to the characteristic function χ_A of A. Therefore, we have the following consequence of Lusin's theorem.

Proposition 5 *If $f : \Omega \to Y$ is μ-measurable, then μ-almost all points of Ω are points of approximate continuity for f.*

Proof: If $\mu(\Omega) < +\infty$, then by Lusin's theorem (Theorem 1), there is an increasing sequence of compact sets $K_n \subseteq \Omega$ such that for each n, $f|K_n$ is continuous and $\mu(\Omega \setminus \cup_n K_n) = 0$. For this case, the result follows from the fact that for each n, μ-almost every point of K_n is a point of density of K_n. The general case follows since Ω has σ-finite measure. □

We also have the following relationship between points of approximate continuity and Lebesgue points.

Proposition 6 *If $f : \Omega \to Y$ is μ-measurable and $a \in \Omega$ is a point of approximate continuity for f and also a Lebesgue point for $\|f\|$ with respect to $\|f(a)\|$, then a is a Lebesgue point for f with respect to $f(a)$.*

Proof: Fix $\varepsilon > 0$. Let $c = \|f(a)\|$ and choose $R > 0$ so that if S is a set in $\mathcal{S}$ with tag a and $S \subseteq B(a,R)$, then

$$\int_S \mid \|f(x)\| - c \mid \ \mu(dx) < \varepsilon \cdot \mu(S),$$

and for $E := \{x \in S : \|f(x) - f(a)\| > \varepsilon\}$, we have $\mu(E) \leq \frac{\varepsilon}{2c+1} \cdot \mu(S)$. Now

$$\int_{E \cap \{\|f\| > c\}} (\|f(x)\| - c) \ \mu(dx) < \varepsilon \cdot \mu(S),$$

whence

$$\int_{E \cap \{\|f\| > c\}} \|f(x)\| \ \mu(dx) \leq 2\varepsilon \cdot \mu(S).$$

Therefore,

$$\begin{aligned} &\int_S \|f(x) - f(a)\| \ \mu(dx) \\ &\leq \int_{S \setminus E} \varepsilon \ \mu(dx) + \int_E 2c \ \mu(dx) + \int_{E \cap \{\|f\| > c\}} \|f(x)\| \ \mu(dx) \leq 4\varepsilon \cdot \mu(S). \end{aligned}$$

Since the choice of $\varepsilon > 0$ is arbitrary, the result follows. □

Remark 7 Proposition 6 also holds for the μ-null set consisting of points $a \in \Omega$ that are Lebesgue points of $\|f\|$ with respect to values different from $\|f(a)\|$.

5 Integration Recall that if $\mu(\Omega) < +\infty$, then a μ-measurable function $f : \Omega \to Y$ is **Bochner integrable** (with respect to μ) if there is a sequence of simple functions $f_n : \Omega \to Y$ with $\lim_n \int_\Omega \|f_n - f\| \, d\mu = 0$. In this case, the sequence of integrals $\int_\Omega f_n \ d\mu$ is Cauchy in Y, and the limit is the Bochner integral of f. Also, f is Bochner integrable if and only if the norm $\|f\|$ is integrable. (See, for example, Chapter II of [5].) We will consider the analogous approximations and integral for the case that $\|f\|$ is integrable on our σ-finite measure space $(X, \mathcal{M}, \mu)$, with the understanding that a simple function is defined on all of X but must vanish off of a set of finite measure. In Theorem 12 of [11], we have established a necessary and sufficient condition for the integrability of a real-valued function, such as $\|f\|$, in terms of approximations by Riemann sums using sets from a μ-a.e. Morse cover. Now, assuming that $\|f\|$ is μ-integrable on X, we show that the Bochner integral of f can also be approximated by the integral of any appropriate simple function using Morse sets.

Theorem 8 *Assume $\|f\|$ is μ-integrable on X, and fix $\varepsilon > 0$. Given $\lambda \geq 1$, and any μ-a.e., λ-Morse cover $\mathcal{S}$ of X, there is a gauge function $\delta : X \to (0,1]$ such that for any finite or countably infinite sequence $\langle S_i(x_i) \rangle$ of δ-fine sets from $\mathcal{S}$ that is μ-exhausting of X, the function $\sum_i f(x_i)\chi_{S_i}$ approximates f in the sense that*

$$\int_X \left\| f(y) - \sum_i f(x_i)\chi_{S_i}(y) \right\| \ \mu(dy) < \varepsilon$$

and the absolute sum of local errors is small; that is,

$$\sum_i \left\| \int_{S_i} f(y) \ \mu(dy) - f(x_i)\mu(S_i) \right\| < \varepsilon. \tag{1}$$

The same approximation holds for the restriction of f to a large open ball Ω about the origin with $\int_{X\setminus\Omega}\|f\|<\varepsilon$, but then there is an $m\in\mathbb{N}$ such that for any $n\geq m$,

$$\begin{aligned}&\left\|\int_X f(y)\,\mu\,(dy)-\sum\nolimits_{i=1}^{n} f(x_i)\,\mu\,(S_i)\right\|\\ \leq &\int_X\left\|f(y)-\sum\nolimits_{i=1}^{n} f(x_i)\chi_{S_i}\,(y)\right\|\,\mu\,(dy)<3\varepsilon.\end{aligned}$$

Proof: Fix $\gamma>0$ so that for each $E\subseteq X$ with $\mu(E)<\gamma$, $\int_E\|f\|\,d\mu<\varepsilon/4$. Let L be the set of points $x\in X$ that are Lebesgue points of f with respect to $f(x)$, and let $A:=X\setminus L$. Since $\mu\,(A)=0$, there is an open set G containing A with $\mu(G)<\gamma$. For each $n\in\mathbb{N}$, we set $A_n=\{x\in A:n-1\leq\|f(x)\|<n\}$. The sets A_n are disjoint and μ-null with union A. For each $n\in\mathbb{N}$, fix an open set G_n with $G\supseteq G_n\supseteq A_n$ and $\mu(G_n)<\varepsilon/\left(n\cdot 2^{n+2}\right)$. For each $x\in A_n$, we choose $\delta(x)<1\wedge\sup\{s:B(x,s)\subseteq G_n\}$. Then for any finite or countably infinite disjoint sequence of δ-fine sets S_i from $\mathcal{S}$ with tags x_i in A, we have

$$\sum_i\|f(x_i)\|\,\mu(S_i)\leq\sum_{n=1}^{\infty}\left(n\sum_{x_i\in A_n}\mu(S_i)\right)\leq\sum_{n=1}^{\infty}\varepsilon\,2^{-n-2}=\varepsilon/4,$$

and the integral of $\|f\|$ over $\cup_i S_i$ is at most $\varepsilon/4$. Set $B(0,-1)=B(0,0)=\varnothing$. If for $n\in\mathbb{N}$, $x\in L\cap(B(0,n)\diagdown B(0,n-1))$, then $B(x,1)\subseteq E_n:=B\,(0,n+1)\diagdown B\,(0,n-2)$, and we choose $\delta(x)$ with $0<\delta(x)\leq 1$ so that every δ-fine $S\in\mathcal{S}$ with tag x satisfies the inequality

$$\int_S\|f(y)-f(x)\|\,\mu(dy)<\frac{\varepsilon\cdot 2^{-n-2}}{[1+\mu\,(E_n)]}\cdot\mu(S).$$

With this choice of the gauge $\delta:X\to(0,1]$, we let $\langle S_i\rangle$ be any finite or countably infinite sequence of δ-fine sets from $\mathcal{S}$ that is μ-exhausting of X. Let I_L be the set of those indices i for which $x_i\in L$, and let I_A be the set of those indices i with $x_i\in A=X\setminus L$. For each $n\in\mathbb{N}$, let I_L^n be those indices in I_L with $x_i\in B(0,n)\backslash B(0,n-1)$. Set $S_A:=\cup_{i\in I_A}S_i(x_i)$ and $S_L:=\cup_{i\in I_L}S_i(x_i)$. Now by the above calculation, $\sum\limits_{i\in I_A}\|f(x_i)\|\,\mu(S_i)\leq\varepsilon/4$, and since $\mu\,(X\backslash(S_L\cup S_A))=0$, we have

$$\int_{X\backslash(S_L\cup S_A)}\|f\|\,d\mu+\int_{S_A}\|f\|\,d\mu+\sum_{i\in I_A}\|f(x_i)\|\,\mu(S_i)\leq\frac{\varepsilon}{2}.$$

Moreover,

$$\begin{aligned}&\int_X\left\|f(y)-\sum\nolimits_i f(x_i)\chi_{S_i}\,(y)\right\|\,\mu\,(dy)\\ \leq&\int_{S_L}\left\|f(y)-\sum\nolimits_{i\in I_L} f(x_i)\chi_{S_i}\,(y)\right\|\,\mu(dy)+\frac{\varepsilon}{2}\\ \leq&\sum_{n=1}^{\infty}\sum_{i\in I_L^n}\int_{S_i}\|f(y)-f(x_i)\|\,\mu(dy)+\frac{\varepsilon}{2}\\ <&\sum_{n=1}^{\infty}\sum_{i\in I_L^n}\frac{\varepsilon\cdot 2^{-n-2}}{[1+\mu\,(E_n)]}\cdot\mu(S(x_i))+\frac{\varepsilon}{2}\leq\varepsilon\cdot\sum_{n=1}^{\infty}2^{-n-2}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

To show that the sum of local errors is small, we note that for each i, $\sum_j f(x_j)\chi_{S_j}(y) = f(x_i)$ on S_i, whence

$$\sum_i \left\| \int_{S_i} f(y)\ \mu(dy) - f(x_i)\mu(S_i) \right\| \leq \sum_i \int_{S_i} \|f(y) - f(x_i)\|\ \mu(dy)$$
$$= \sum_i \int_{S_i} \left\| f(y) - \sum_j f(x_j)\chi_{S_j}(y) \right\|\ \mu(dy)$$
$$= \int_X \left\| f(y) - \sum_j f(x_j)\chi_{S_j}(y) \right\|\ \mu(dy) < \varepsilon.$$

For the last inequality, choose $m \in \mathbb{N}$ so that $\mu(\Omega \setminus \cup_{i \leq m} S_i) < \gamma$, and the proof follows essentially as before. $\square$

Remark 9 Note that our simple functions are not formed using a partition of X, as is usual for the Henstock–Kurzweil and McShane integrals; they are, however, defined on all of X. Also note that partitions are allowed in Theorem 8. That is, suppose a Cartesian coordinate system has been imposed on X and $I \subset X$ is a bounded open interval. In McShane integration over I, one takes a partition $\langle S_i \rangle_{i=1}^N$ to be intervals in I with $\cup_{i=1}^N S_i = I$. If these intervals are mutually disjoint, satisfy the λ-regularity condition, and are δ-fine, then they are examples of the covering sets in Theorem 8.

Definition 10 *A function $G : \mathcal{M} \to Y$ is called **countably additive** if $G(\cup_i E_i) = \lim_{n\to\infty} \sum_i^n G(E_i)$ in norm for every ordering of any infinite, pairwise disjoint sequence $\langle E_i \rangle$ from $\mathcal{M}$. It is also required that $G(\varnothing) = 0$, whence G is finitely additive.*

Corollary 11 *Let μ be a Radon measure on $(X, \mathcal{M})$. Let $\mathcal{S}$ be a μ-a.e., λ-Morse cover of X such that for each $S \in \mathcal{S}$, $\mu(S \setminus S^\circ) = 0$, where S° is the interior of S. Suppose $G : \mathcal{M} \to Y$ is countably additive and has the additional properties that $G(E) = 0$ for each $E \in \mathcal{M}$ with $\mu(E) = 0$, and*

$$M := \sup \left\{ \sum_i \|G(S_i)\| : \langle S_i \rangle \subset \mathcal{S},\quad \langle S_i \rangle\ \mu\text{-exhausting of } X \right\} < +\infty.$$

Also suppose that there is a μ-measurable $f : X \to Y$ such that for any $\varepsilon > 0$ there is a gauge $\delta : X \to (0,1]$ with the property that for any δ-fine sequence $\langle S_i(x_i) \rangle$ in $\mathcal{S}$ that is μ-exhausting of X we have

$$\sum_i \|f(x_i)\mu(S_i) - G(S_i)\| < \varepsilon. \tag{2}$$

Then f is μ-integrable and $\int_X f\, d\mu = G(X)$.

Proof: Fix $\varepsilon > 0$. Also, fix $\langle T_j \rangle \subset \mathcal{S}$ so that $\langle T_j \rangle$ is a sequence that is μ-exhausting of X and $\sum_j \|G(T_j)\| > M - \varepsilon/2$. Let $\Omega = \cup_j T_j^\circ$. We choose a gauge $\delta : X \to (0,1]$ so that for any δ-fine sequence $\langle S_i \rangle$ in $\mathcal{S}$ that is μ-exhausting of X, Equation 2 holds with ε replaced by $\varepsilon/2$ and if $x \in T_j^\circ$ for some j, then $B(x, \delta(x)) \subseteq T_j^\circ$. The collection $\mathcal{S}_\Omega$ of δ-fine sets in $\mathcal{S}$ with tags in Ω is a μ-a.e., Morse cover of Ω. Let $\langle S_i \rangle$ be a δ-fine sequence in $\mathcal{S}_\Omega$ such that $\langle S_i \rangle$ is μ-exhausting of Ω and therefore of X. We have for any ordering,

$$\left| \sum_i \|f(x_i)\|\mu(S_i) - \sum_i \|G(S_i)\| \right| \leq \sum_i |\ \|f(x_i)\mu(S_i)\| - \|G(S_i)\|\ |$$
$$\leq \sum_i \|f(x_i)\mu(S_i) - G(S_i)\| < \frac{\varepsilon}{2}.$$

Moreover, for each j, $\mu\left(T_j \setminus \bigcup_{S_i \subseteq T_j} S_i\right) = 0$, so

$$\begin{aligned} M &\geq \sum_i \|G(S_i)\| = \sum_j \sum_{S_i \subseteq T_j} \|G(S_i)\| \\ &\geq \sum_j \left\| G\left(\bigcup_{S_i \subseteq T_j} S_i\right)\right\| = \sum_j \|G(T_j)\| > M - \frac{\varepsilon}{2}. \end{aligned}$$

Therefore,

$$\left|\sum_i \|f(x_i)\|\mu(S_i) - M\right| < \varepsilon.$$

By Theorem 12 in [11], $\|f\|$ is μ-integrable on Ω, and therefore on X, whence f is μ-integrable on X.

To show that $\int_X f\,d\mu = G(X)$, we fix a gauge δ so that Equations 1 and 2 hold for any sequence $\langle S_i(x_i)\rangle$ of δ-fine sets from $\mathcal{S}$ that is μ-exhausting of X. Then given such a sequence, we have

$$\begin{aligned} &\left\| G(X) - \int_X f\,d\mu \right\| \\ &= \left\| \sum_i \left(G(S_i) - \int_{S_i} f\,d\mu \right) \right\| \leq \sum_i \left\| G(S_i) - \int_{S_i} f\,d\mu \right\| \\ &\leq \sum_i \left\| \int_{S_i} f\,d\mu - f(x_i)\mu(S_i) \right\| + \sum_i \|f(x_i)\mu(S_i) - G(S_i)\| < 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, $\int_X f\,d\mu = G(X)$. □

References

[1] A.S. Besicovitch, A general form of the covering principle and relative differentiation of additive functions (I), (II), *Proc. Cambridge Phil. Soc.* **41**(1945) 103–110, **42**(1946) 1–10.

[2] J. Bliedtner and P.A. Loeb, The optimal differentiation basis and liftings of L^∞, *Trans. Amer. Math. Soc.* **352**(2000) 4693–4710.

[3] L. Di Piazza and V. Marraffa, An equivalent definition of the vector-valued McShane integral by means of partitions of unity, *Studia Math.* **151**(2002) 175–185.

[4] L. Di Piazza and K. Musiał, A characterization of variationally McShane integrable Banach-space valued functions, *Illinois J. Math.* **45**(2001) 279–289.

[5] J. Diestel and J.J. Uhl, Jr., *Vector Measures*, American Mathematical Society, Providence, 1977.

[6] N. Dunford and J. Schwartz, *Linear Operators, Part I*, Interscience Publishers Inc., New York, 1957.

[7] D.H. Fremlin, Measurable functions and almost continuous functions, *Manuscripta Math.*, **33**(1981) 387–405.

[8] ______, The generalized McShane integral, *Illinois J. Math.* **39**(1995) 39–67.

[9] Z. Füredi and P.A. Loeb, On the best constant for the Besicovitch covering theorem, *Proc. Amer. Math. Soc.* **121**(1994) 1063–1073.

[10] J. Kupka and K. Prikry, The measurability of uncountable unions, *Amer. Math. Monthly* **91**(1984) 85–97.

[11] P.A. Loeb and E. Talvila, Covering theorems and Lebesgue integration, *Sci. Math. Jpn.* **53**(2001) 209–221.

[12] A.P. Morse, Perfect blankets, *Trans. Amer. Math. Soc.* **61**(1947) 418–442.

[13] J.C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, Berlin, 1971.

[14] J.M.A.M. van Neerven, Approximating Bochner integrals by Riemann sums, *Indag. Math. (N.S.)* **13**(2002), 197–208.

Peter A. Loeb
Department of Mathematics, University of Illinois
1409 West Green Street, Urbana, Illinois 61801, U.S.A.
e-mail: loeb@math.uiuc.edu

Erik Talvila
Department of Mathematics and Statistics
University College of the Fraser Valley
Abbotsford, BC, Canada V2S 7M8
e-mail: Erik.Talvila@ucfv.ca