ON THE CATEGORICAL ALGEBRAS OF FIRST-ORDER LOGIC*

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Abstract. A category **Fol** of sets of formulas of first-order languages with finitary relations and with equality is constructed. An adjunction $\langle F, U, \eta, \epsilon\rangle : \mathbf{HSet} \to \mathbf{Fol}$, where **HSet** denotes the category of families of sets indexed by subsets of the natural numbers "respecting inclusions" is then obtained. It gives rise to an algebraic theory **T** over **HSet**. It is then shown that the Eilenberg-Moore category of **T**-algebras in **HSet** has a subcategory isomorphic to the category $\overrightarrow{\mathrm{Lf}_\omega}$ corresponding to the variety Lf_ω of all ω-dimensional locally finite cylindric algebras. Moreover, its subcategory with objects all largest locally finite subalgebras of full ω-dimensional cylindric set algebras is the category of algebras that was used in the categorical algebraization of a version of the institution of first-order logic without terms presented in previous work by the author.

1 Introduction This paper continues the exploration of the connections between the algebraization of well-known logics using the modern categorical algebraization process (see [13, 14, 15, 16]) and their algebraization using the traditional methods. The categorical process is a generalization using the institution framework [6, 7], of the very well-known universal algebraic algebraization framework of Blok and Pigozzi (see [2, 3]) that can handle more effectively logics with multiple signatures and quantifiers. The hope and the aim is that, by showing that the modern method leads to the same or equivalent results with the traditional methods and by comparing the two processes, many of the advantages and disadvantages of each will be revealed and this will eventually lead to an improved version of the modern technique and to its wider applicability to less known logical systems. The first step in this direction was accomplished with the categorical algebraization of equational logic in [17] and the subsequent investigation [18] of the relation of the category of algebras on which the equivalent algebraic institution semantics was based with classes of algebras of clones that had been given in the literature using more traditional constructions. The next step is undertaken in the present work where the relation between the category of algebras used in the categorical algebraization of first-order logic without terms of [19] is compared with the class of all locally finite ω-dimensional cylindric algebras (see [8, 9, 10]) and a subclass of the class of all full cylindric set algebras of dimension ω. These classes were discovered during the efforts to create an equivalent algebraic semantics for first-order logic without terms in the traditional approach. The algebraic theory that was used in [19] corresponds to a variety strictly larger than the variety generated by the locally finite cylindric algebras but its subcategory that was used as the category of algebras on which the algebraization was based is a subcategory of the category of locally finite cylindric algebras of dimension ω. It should be noted that that the word "sub", where used above, should be

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taken in the categorical sense of the existence of an injective functor, not in the literally sense of an embedding.

2 The Algebraic Theory We recall some of the basic constructions introduced in [19]. We refer the reader to [4] and [11] for all unexplained categorical notation and to [12] for more details on the correspondence between algebraic theories and varieties of algebras.

In what follows, by **Set** will be denoted the category of all small sets, by ω the set of natural numbers, $N \subseteq_{\mathrm{f}} \omega$ will mean that N is a finite subset of ω and $\mathcal{P}_{\mathrm{f}}(\omega)$ will denote the set of all finite subsets of ω.

By a **hierarchy of sets** or, simply, an **h-set** A, we mean a family of sets $A = \{A_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, such that $A_N \cap A_M = A_{N\cap M}$, for every $N, M \subseteq_{\mathrm{f}} \omega$. By a **morphism of h-sets** or, simply, an **h-set morphism** $f : A \to B$, we mean a family of set maps $f = \{f_N : A_N \to B_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, such that the following diagram commutes, for every $N \subseteq M \subseteq_{\mathrm{f}} \omega$,

$$\begin{tikzcd} A_M \arrow[r, "f_M"] & B_M \\ A_N \arrow[u, "i"] \arrow[r, "f_N"'] & B_N \arrow[u, "i"'] \end{tikzcd}$$

where by $i : A_N \hookrightarrow A_M$ and $i : B_N \hookrightarrow B_M$ we denote the inclusion maps.

Given two chain set morphisms $f : A \to B$ and $g : B \to C$ we define their **composite** $gf : A \to C$ to be the collection of maps $gf = \{g_N f_N : A_N \to C_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\}$. With this composition the collection of h-sets with h-set morphisms between them forms a category. It is called the **category of h-sets** and denoted by **HSet**.

The construction of this category as the underlying category of the institution of first-order logic was inspired by a well known formalization of first order logic in which all the relation symbols are ranked (see, e.g., [1]).

By $\mathcal{L}$ is denoted the set of symbols $\{\neg, \wedge\} \cup \{\exists_k : k \in \omega\}$, which will be used as connectives and quantifiers, respectively, in the construction of the formulas below. Given a set X, by $\overline{X}$ will be denoted an isomorphic copy of X constructed in some canonical way. $\overline{x}$ will denote the copy of $x \in X$ in the set $\overline{X}$.

Given $X \in |\mathbf{HSet}|$, the **h-set of X-formulas**

$$\mathrm{Fm}_{\mathcal{L}}(X) = \{\mathrm{Fm}_{\mathcal{L}}(X)_N : N \in \mathcal{P}_{\mathrm{f}}(\omega)\} \in |\mathbf{HSet}|$$

is defined by letting $\mathrm{Fm}_{\mathcal{L}}(X)_N$ be the smallest set satisfying $\mathrm{v}_i \approx \mathrm{v}_j \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for all $i, j \in N$, $\overline{x} \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for every $x \in X_N$, $\neg\phi, \phi_1 \wedge \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, and, finally, $\exists_k \phi \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, for every $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N\cup\{k\}}$.

Given two h-sets X and Y, any h-set morphism f from X into the h-set $\mathrm{Fm}_{\mathcal{L}}(Y)$ may be extended to an h-set morphism f^* from $\mathrm{Fm}_{\mathcal{L}}(X)$ into $\mathrm{Fm}_{\mathcal{L}}(Y)$. This is done as follows: For $X, Y \in |\mathbf{HSet}|, f : X \to \mathrm{Fm}_{\mathcal{L}}(Y) \in \mathrm{Mor}(\mathbf{HSet})$, define $f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$, with $f^*_N : \mathrm{Fm}_{\mathcal{L}}(X)_N \to \mathrm{Fm}_{\mathcal{L}}(Y)_N$, for every $N \subseteq_{\mathrm{f}} \omega$, by recursion on the structure of X-formulas by $f^*_N(\mathrm{v}_i \approx \mathrm{v}_j) = \mathrm{v}_i \approx \mathrm{v}_j$, for all $i, j \in N$, $f^*_N(\overline{x}) = f_N(x)$, for every $x \in X_N$, $f^*_N(\neg\phi) = \neg f^*_N(\phi), f^*_N(\phi_1 \wedge \phi_2) = f^*_N(\phi_1) \wedge f^*_N(\phi_2)$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_N$, and, finally, $f^*_N(\exists_k \phi) = \exists_k f^*_{N\cup\{k\}}(\phi)$, for every $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{N\cup\{k\}}$.

In the sequel, we write $f : X \rightharpoondown Y$ to denote an **HSet**-morphism $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$. Given two such maps $f : X \rightharpoondown Y, g : Y \rightharpoondown Z$, their **composition** $g \circ f : X \rightharpoondown Z$ is defined to be the **HSet**-morphism

$$g \circ f = g^* f.$$

It is shown in [19] that the composition $\circ$ is associative.

Now define $j_X : X \rightharpoondown X$, given by $j_{X_N} : X_N \to \mathrm{Fm}_{\mathcal{L}}(X)_N$, with

$$j_{X_N}(x) = \overline{x}, \quad \text{for all } x \in X_N.$$

Given $f : X \rightharpoondown Y$ and $g : Z \rightharpoondown X$ we have $f \circ j_X = f$ and $j_X \circ g = g$.

Therefore **Fol**, having collection of objects $|\mathbf{HSet}|$ and collection of morphisms

$$\mathbf{Fol}(X,Y) = \{f : X \rightharpoondown Y : f \in \mathbf{HSet}(X, \mathrm{Fm}_{\mathcal{L}}(Y))\},$$

for all $X, Y \in |\mathbf{HSet}|$, with composition $\circ$ and X-identity j_X, is a category.

Now, define a functor $F : \mathbf{HSet} \to \mathbf{Fol}$ by $F(X) = X$, for all $X \in |\mathbf{HSet}|$, and, given $f : X \to Y \in \mathrm{Mor}(\mathbf{HSet})$, $F(f) = j_Y f : X \rightharpoondown Y$.

Next, define a functor $U : \mathbf{Fol} \to \mathbf{HSet}$ by $U(X) = \mathrm{Fm}_{\mathcal{L}}(X)$, for all $X \in |\mathbf{Fol}|$, and, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Fol})$, $U(f) = f^* : \mathrm{Fm}_{\mathcal{L}}(X) \to \mathrm{Fm}_{\mathcal{L}}(Y)$, the extension of $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$ to X-formulas.

Finally, define natural transformations $\eta : I_{\mathbf{HSet}} \to UF$ by $\eta_X : X \to \mathrm{Fm}_{\mathcal{L}}(X)$, with $\eta_X = j_X$, for all $X \in |\mathbf{HSet}|$, and $\epsilon : FU \to I_{\mathbf{Fol}}$ by $\epsilon_X : \mathrm{Fm}_{\mathcal{L}}(X) \rightharpoondown X$, with $\epsilon_X = i_{\mathrm{Fm}_{\mathcal{L}}(X)}$, for all $X \in |\mathbf{Fol}|$.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathrm{Fm}_{\mathcal{L}}(X) \arrow[d, "(j_Y f)^*"] \\
Y \arrow[r, "\eta_Y"'] & \mathrm{Fm}_{\mathcal{L}}(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fm}_{\mathcal{L}}(X) \arrow[r, "\epsilon_X"] \arrow[d, "j_{\mathrm{Fm}_{\mathcal{L}}(Y)} f^*"'] & X \arrow[d, "f"] \\
\mathrm{Fm}_{\mathcal{L}}(Y) \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}$$

The functors $F : \mathbf{HSet} \to \mathbf{Fol}$ and $U : \mathbf{Fol} \to \mathbf{HSet}$ are adjoints with unit η and counit ϵ.

To create the algebraic theory, we set $T = UF$ and $\mu = U\epsilon_F$. $T : \mathbf{HSet} \to \mathbf{HSet}$ is a functor, since it is the composite of two functors, and $\mu : TT \to T$ is a natural transformation, since ϵ is a natural transformation. Furthermore, the triangular identities of the adjunction induce the commutativity of the following diagrams, that are the prerequisites for $\mathbf{T} = \langle T, \eta, \mu \rangle$ to be an algebraic theory in **HSet**.

$$\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[dr, "i_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "T(\eta)"'] \arrow[dl, "i_T"] \\
& T &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTT \arrow[r, "T(\mu)"] \arrow[d, "\mu_T"'] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"'] & T
\end{tikzcd}$$

Moreover, there exists a unique functor $K : \mathbf{HSet_T} \to \mathbf{Fol}$ from the Kleisli category of the theory $\mathbf{T}$ to **Fol**, called the Kleisli comparison functor of the adjunction, that makes the F- and the U-paths of the following triangles commute

$$\begin{tikzcd}
\mathbf{HSet_T} \arrow[rr, "K"] & & \mathbf{Fol} \\
& \mathbf{HSet} \arrow[ul, "F_{\mathbf{T}}"] \arrow[ur, "F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{HSet_T} \arrow[rr, "K"] \arrow[dr, "U_{\mathbf{T}}"'] & & \mathbf{Fol} \arrow[dl, "U"] \\
& \mathbf{HSet} &
\end{tikzcd}$$

Since the Kleisli category of an algebraic theory has, by definition, as objects the same objects with the underlying category of the theory and as morphisms from an object X to an object Y all the morphisms in the underlying category from X to $T(Y)$, with composition

$\circ_K$ given by $g \circ_K f = \mu_Y T(g) f$, for all $f : X \to T(Y), g : Y \to T(Z) \in \mathrm{Mor}(\mathbf{HSet})$, it is easy to see that in this case $\mathbf{HSet_T} = \mathbf{Fol}$ and $K = I_{\mathbf{Fol}}$. Hence **Fol** is the category of all free algebras of the algebraic theory **T** in **HSet**.

A **T**-algebra $\mathbf{X} = \langle X, \xi \rangle$ in **HSet** is now a pair consisting of an h-set X together with an **HSet**-morphism $\xi : \mathrm{Fm}_{\mathcal{L}}(X) \to X$, such that the two diagrams on the left below commute

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[dr, "i_X"'] & \mathrm{Fm}_{\mathcal{L}}(X) \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fm}_{\mathcal{L}}(\mathrm{Fm}_{\mathcal{L}}(X)) \arrow[r, "(j_X\xi)^*"] \arrow[d, "i^*_{\mathrm{Fm}_{\mathcal{L}}(X)}"'] & \mathrm{Fm}_{\mathcal{L}}(X) \arrow[d, "\xi"] \\ \mathrm{Fm}_{\mathcal{L}}(X) \arrow[r, "\xi"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fm}_{\mathcal{L}}(X) \arrow[r, "(j_Y h)^*"] \arrow[d, "\xi"'] & \mathrm{Fm}_{\mathcal{L}}(Y) \arrow[d, "\zeta"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

The h-set X is called the carrier of $\mathbf{X}$ and the morphism ξ is called the structure map of $\mathbf{X}$. Moreover, given two **T**-algebras $\mathbf{X} = \langle X, \xi \rangle$ and $\mathbf{Y} = \langle Y, \zeta \rangle$, a **T**-algebra homomorphism $h : \mathbf{X} \to \mathbf{Y}$ is an **HSet**-morphism $h : X \to Y$, such that the third diagram above commutes. The category with collection of objects the collection of all **T**-algebras and with morphisms all **T**-algebra homomorphisms between them is known as the Eilenberg-Moore category of **T**-algebras in **HSet** and is denoted by $\mathbf{HSet^T}$.

3 Connections with Cylindric Algebras The definition of a cylindric algebra from [8] is recalled next. For motivation and further details on the structure and other issues of the theory of cylindric algebras see [8] and [9].

Definition 1 *A* cylindric algebra of dimension α, *where α is any ordinal, is an algebra* $\mathbf{A} = \langle A, +, \cdot, -, 0, 1, \mathrm{c}_\kappa, \mathrm{d}_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$, *where* $0, 1, \mathrm{d}_{\kappa\lambda}$ *are nullary,* $-, \mathrm{c}_\kappa$ *are unary and* $+, \cdot$ *are binary satisfying the following postulates, for all* $x, y \in A, \kappa, \lambda, \mu < \alpha$,

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ *is a Boolean algebra,*
2. $\mathrm{c}_\kappa 0 = 0$
3. $x \leq \mathrm{c}_\kappa x$, *i.e.,* $x + \mathrm{c}_\kappa x = \mathrm{c}_\kappa x$,
4. $\mathrm{c}_\kappa(x \cdot \mathrm{c}_\kappa y) = \mathrm{c}_\kappa x \cdot \mathrm{c}_\kappa y$
5. $\mathrm{c}_\kappa \mathrm{c}_\lambda x = \mathrm{c}_\lambda \mathrm{c}_\kappa x$
6. $\mathrm{d}_{\kappa\kappa} = 1$
7. *if* $\kappa \neq \lambda, \mu$, *then* $\mathrm{d}_{\lambda\mu} = \mathrm{c}_\kappa(\mathrm{d}_{\lambda\kappa} \cdot \mathrm{d}_{\kappa\mu})$
8. *if* $\kappa \neq \lambda$, *then* $\mathrm{c}_\kappa(\mathrm{d}_{\kappa\lambda} \cdot x) \cdot \mathrm{c}_\kappa(\mathrm{d}_{\kappa\lambda} \cdot -x) = 0$.

The class of all cylindric algebras is denoted by CA *and the class of all cylindric algebras of dimension* α *by* CA_α *The elements* $\mathrm{d}_{\kappa\lambda}$ *are called the* diagonal elements *and the operations* c_κ *are called the* cylindrifications.

By the **dimension set** of $x \in A$, in symbols $\Delta^{\mathbf{A}} x$, or simply Δx, is meant the set of all κ for which $\mathrm{c}_\kappa x \neq x$.

A cylindric algebra $\mathbf{A}$ is said to be **locally finite of dimension** α if it is of dimension α and $|\Delta x| < \omega$, for every $x \in A$. The class of all locally finite cylindric algebras of dimension α is denoted by Lf_α. The corresponding category is denoted by $\overrightarrow{\mathrm{Lf}_\alpha}$.

The goal in what follows is to construct an injection from $\overrightarrow{\mathrm{Lf}_\omega}$ into $\mathbf{HSet^T}$ and to investigate the relationship between the two categories.

Let $\mathbf{A} = \langle A, +, \cdot, -, 0, 1, \mathrm{c}_\kappa, \mathrm{d}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ be a locally finite cylindric algebra of dimension ω. Let $A^\dagger = \{A^\dagger_N : N \in \mathcal{P}_\mathrm{f}(\omega)\}$ be defined by setting $A^\dagger_N = \{a \in A : \Delta a \subseteq N\}$. We then have

$$\begin{array}{rll} a \in A^\dagger_N \cap A^\dagger_M & \text{iff} & \Delta a \subseteq N \text{ and } \Delta a \subseteq M \\ & \text{iff} & \Delta a \subseteq N \cap M \\ & \text{iff} & a \in A^\dagger_{N\cap M}. \end{array}$$

Thus $A^\dagger_N \cap A^\dagger_M = A^\dagger_{N\cap M}$ and $A^\dagger$ is an h-set. Next, define a map $\xi_{\mathbf{A}^\dagger} : \mathrm{Fm}_\mathcal{L}(A^\dagger) \to A^\dagger$ by recursion on the structure of an $A^\dagger$-formula, by letting $\xi_{\mathbf{A}^\dagger{}_N} : \mathrm{Fm}_\mathcal{L}(A^\dagger)_N \to A^\dagger_N$ be given by

- $\xi_{\mathbf{A}^\dagger{}_N}(\mathrm{v}_i \approx \mathrm{v}_j) = \mathrm{d}_{ij}$,
- $\xi_{\mathbf{A}^\dagger{}_N}(\overline{a}) = a$, for all $a \in A^\dagger_N$,
- $\xi_{\mathbf{A}^\dagger{}_N}(\neg\phi) = -\xi_{\mathbf{A}^\dagger{}_N}(\phi)$,
- $\xi_{\mathbf{A}^\dagger{}_N}(\phi_1 \wedge \phi_2) = \xi_{\mathbf{A}^\dagger{}_N}(\phi_1) \cdot \xi_{\mathbf{A}^\dagger{}_N}(\phi_2)$,
- $\xi_{\mathbf{A}^\dagger{}_N}(\exists_k\phi) = \mathrm{c}_k\xi_{\mathbf{A}^\dagger{}_{N\cup\{k\}}}(\phi)$.

It is not difficult to verify that $\xi_{\mathbf{A}^\dagger} = \{\xi_{\mathbf{A}^\dagger{}_N} : N \in \mathcal{P}_\mathrm{f}(\omega)\}$ is an h-set morphism.

The following lemma shows that the pair $\langle A^\dagger, \xi_{\mathbf{A}^\dagger}\rangle$ forms a $\mathbf{T}$-algebra in $\mathbf{HSet}$.

Lemma 2 *Let* $\mathbf{A} \in \overrightarrow{\mathrm{Lf}_\omega}$. *Then* $\mathbf{A}^\dagger = \langle A^\dagger, \xi_{\mathbf{A}^\dagger}\rangle \in |\mathbf{HSet}^\mathbf{T}|$.

Proof:
It suffices to show that the following diagrams commute:

$$\begin{tikzcd} A^\dagger \arrow[r, "\eta_{A^\dagger}"] \arrow[dr, "i_{A^\dagger}"'] & \mathrm{Fm}_\mathcal{L}(A^\dagger) \arrow[d, "\xi_{\mathbf{A}^\dagger}"] \\ & A^\dagger \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fm}_\mathcal{L}(\mathrm{Fm}_\mathcal{L}(A^\dagger)) \arrow[r, "(\eta_{A^\dagger}\xi_{\mathbf{A}^\dagger})^*"] \arrow[d, "i^*_{\mathrm{Fm}_\mathcal{L}(A^\dagger)}"'] & \mathrm{Fm}_\mathcal{L}(A^\dagger) \arrow[d, "\xi_{\mathbf{A}^\dagger}"] \\ \mathrm{Fm}_\mathcal{L}(A^\dagger) \arrow[r, "\xi_{\mathbf{A}^\dagger}"'] & A^\dagger \end{tikzcd}$$

Verification of the commutativity of the triangle is straightforward and will be omitted. For the rectangle induction on the structure of an $\mathrm{Fm}_\mathcal{L}(A^\dagger)$-term is needed. Only the cases of the terms $\mathrm{v}_i \approx \mathrm{v}_j$ and $\exists_k\phi$ will be presented. The remaining cases may be handled similarly. We have

$$\begin{array}{rcl} \xi_{\mathbf{A}^\dagger{}_N}((\eta_{A^\dagger}\xi_{\mathbf{A}^\dagger})^*_N(\mathrm{v}_i \approx \mathrm{v}_j)) & = & \xi_{\mathbf{A}^\dagger{}_N}(\overline{d_{ij}}) \\ & = & d_{ij} \\ & = & \xi_{\mathbf{A}^\dagger{}_N}(\mathrm{v}_i \approx \mathrm{v}_j) \\ & = & \xi_{\mathbf{A}^\dagger{}_N}(i^*_{\mathrm{Fm}_\mathcal{L}(A^\dagger)_N}(\mathrm{v}_i \approx \mathrm{v}_j)). \end{array}$$

and

$$\begin{array}{rcl} \xi_{\mathbf{A}^\dagger{}_N}((\eta_{A^\dagger}\xi_{\mathbf{A}^\dagger})^*_N(\exists_k\phi)) & = & \xi_{\mathbf{A}^\dagger{}_N}(\exists_k(\eta_{A^\dagger}\xi_{\mathbf{A}^\dagger})^*_{N\cup\{k\}}(\phi)) \\ & = & \mathrm{c}_k\xi_{\mathbf{A}^\dagger{}_{N\cup\{k\}}}((\eta_{A^\dagger}\xi_{\mathbf{A}^\dagger})^*_{N\cup\{k\}}(\phi)) \\ & = & \mathrm{c}_k\xi_{\mathbf{A}^\dagger{}_{N\cup\{k\}}}(i^*_{\mathrm{Fm}_\mathcal{L}(A^\dagger)_{N\cup\{k\}}}(\phi)) \\ & = & \xi_{\mathbf{A}^\dagger{}_N}(\exists_k i^*_{\mathrm{Fm}_\mathcal{L}(A^\dagger)_{N\cup\{k\}}}(\phi)) \\ & = & \xi_{\mathbf{A}^\dagger{}_N}(i^*_{\mathrm{Fm}_\mathcal{L}(A^\dagger)_N}(\exists_k\phi)). \end{array}$$

■

Suppose, next, that $\mathbf{A},\mathbf{B}\in \mathrm{Lf}_\omega$ and $h:\mathbf{A}\to\mathbf{B}\in\overrightarrow{\mathrm{Lf}_\omega}(\mathbf{A},\mathbf{B})$. Define $h^\dagger=\{h^\dagger_N : N\subseteq_{\mathrm{f}}\omega\}$, by $h^\dagger_N = h\restriction_{A^\dagger_N}$. It is now shown that $h^\dagger\in\mathbf{HSet}^{\mathbf{T}}(\mathbf{A}^\dagger,\mathbf{B}^\dagger)$, i.e., that the following rectangle commutes:

$$\begin{tikzcd}
\mathrm{Fm}_{\mathcal{L}}(A^\dagger) \arrow[r, "(\eta_{B^\dagger}h^\dagger)^*"] \arrow[d, "\xi_{\mathbf{A}^\dagger}"'] & \mathrm{Fm}_{\mathcal{L}}(B^\dagger) \arrow[d, "\xi_{\mathbf{B}^\dagger}"] \\
A^\dagger \arrow[r, "h^\dagger"'] & B^\dagger
\end{tikzcd}$$

We only verify the case of the $A^\dagger$-terms $\overline{a},\phi_1\wedge\phi_2$ and $\exists_k\phi$.

$$\begin{aligned}
\xi_{\mathbf{B}^\dagger{}_N}((\eta_{B^\dagger}h^\dagger)^*_N(\overline{a})) &= \xi_{\mathbf{B}^\dagger{}_N}(\overline{h^\dagger_N(a)})\\
&= h^\dagger_N(a)\\
&= h^\dagger_N(\xi_{\mathbf{A}^\dagger{}_N}(\overline{a})),
\end{aligned}$$

$$\begin{aligned}
\xi_{\mathbf{B}^\dagger{}_N}((\eta_{B^\dagger}h^\dagger)^*_N(\phi_1\wedge\phi_2)) &= \xi_{\mathbf{B}^\dagger{}_N}((\eta_{B^\dagger}h^\dagger)^*_N(\phi_1)\wedge(\eta_{B^\dagger}h^\dagger)^*_N(\phi_2))\\
&= \xi_{\mathbf{B}^\dagger{}_N}((\eta_{B^\dagger}h^\dagger)^*_N(\phi_1))\cdot\xi_{\mathbf{B}^\dagger{}_N}((\eta_{B^\dagger}h^\dagger)^*_N(\phi_2))\\
&= h^\dagger_N(\xi_{\mathbf{A}^\dagger{}_N}(\phi_1))\cdot h^\dagger_N(\xi_{\mathbf{A}^\dagger{}_N}(\phi_2))\\
&= h^\dagger_N(\xi_{\mathbf{A}^\dagger{}_N}(\phi_1)\cdot\xi_{\mathbf{A}^\dagger{}_N}(\phi_2))\\
&= h^\dagger_N(\xi_{\mathbf{A}^\dagger{}_N}(\phi_1\wedge\phi_2)),
\end{aligned}$$

$$\begin{aligned}
\xi_{\mathbf{B}^\dagger{}_N}((\eta_{B^\dagger}h^\dagger)^*_N(\exists_k\phi)) &= \xi_{\mathbf{B}^\dagger{}_N}(\exists_k(\eta_{B^\dagger}h^\dagger)^*_{N\cup\{k\}}(\phi))\\
&= \mathrm{c}_k\xi_{\mathbf{B}^\dagger{}_{N\cup\{k\}}}((\eta_{B^\dagger}h^\dagger)^*_{N\cup\{k\}}(\phi))\\
&= \mathrm{c}_k h^\dagger_{N\cup\{k\}}(\xi_{\mathbf{A}^\dagger{}_{N\cup\{k\}}}(\phi))\\
&= h^\dagger_N(\exists_k\xi_{\mathbf{A}^\dagger{}_{N\cup\{k\}}}(\phi))\\
&= h^\dagger_N(\xi_{\mathbf{A}^\dagger{}_N}(\exists_k\phi)).
\end{aligned}$$

In this way, an embedding $P:\overrightarrow{\mathrm{Lf}_\omega}\to\mathbf{HSet}^{\mathbf{T}}$ may be defined by

$$P(\mathbf{A})=\mathbf{A}^\dagger,\quad \text{for every}\quad \mathbf{A}\in\mathrm{Lf}_\omega,$$

and, given $h\in\overrightarrow{\mathrm{Lf}_\omega}(\mathbf{A},\mathbf{B}), P(h)\in\mathbf{HSet}^{\mathbf{T}}(\mathbf{A}^\dagger,\mathbf{B}^\dagger)$, by $P(h)=h^\dagger=\{h\restriction_{A^\dagger_N}: N\subseteq_{\mathrm{f}}\omega\}$.

It is clear that $P(\overrightarrow{\mathrm{Lf}_\omega})\neq\mathbf{HSet}^{\mathbf{T}}$, since, for instance, the free $\mathbf{T}$-algebra in $\mathbf{HSet}$ with universe the h-set $\mathrm{Fm}_{\mathcal{L}}(\emptyset)$, with $\emptyset_N=\emptyset$, for all $N\subseteq_{\mathrm{f}}\omega$, does not satisfy

$$\xi_{\mathbf{Fm}_{\mathcal{L}}(\emptyset)_N}(\mathrm{v}_i\approx\mathrm{v}_i)=\xi_{\mathbf{Fm}_{\mathcal{L}}(\emptyset)_N}(\mathrm{v}_j\approx\mathrm{v}_j),\quad \text{for } i,j\in N, i\neq j.$$

whereas every $\mathbf{T}$-algebra of the form $P(\mathbf{A})$ for some $\mathbf{A}\in\mathrm{Lf}_\omega$, does satisfy this identity.

The above remark also depicts one of the differences of the modern systematic universal and categorical algebraization processes as compared to the more traditional ad-hoc process. One does not need to discover the identities that define the equivalent category of algebras as long as an appropriate adjunction generates a supercategory. Then a subcategory can be specified that provides the required deductive equivalence with the deductive mechanism of the original π-institution. Quite similar to this mechanism is the mechanism generating the equivalent algebraic semantics via the Leibniz operator in the Blok-Pigozzi theory [2] and the generation of the equivalent algebraic semantics in the semantical algebraization process of the Budapest school [1]. The main difference is that the algebraic signature is fixed in advance in both of these settings whereas in the present setting the algebraic clone is generated directly out of a formalization of the logic that includes its substitutions or signature morphisms.

In [19] a special full subcategory $\mathbf{Q} \leq \mathbf{HSet}^{\mathbf{T}}$ was used as the category of algebras of an equivalent algebraic institution semantics for the institution of first-order logic without terms. The objects of that category were termed **relation algebras** and were defined as follows: Given a set A, $\mathrm{Rel}(A)$ denotes the h-set whose N-th level $\mathrm{Rel}_N(A)$ consists of all relations $r \subseteq A^\omega$ that depend only on the individual variables indexed by elements in N. Given such a set A, we define $\mathbf{A}^* = \langle \mathrm{Rel}(A), \xi_A \rangle$, where $\xi_A : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A)) \to \mathrm{Rel}(A) \in \mathrm{Mor}(\mathbf{HSet})$ is determined by $\xi_{A_N} : \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N \to \mathrm{Rel}_N(A)$, defined by recursion on the structure of $\mathrm{Rel}(A)$-formulas as follows:

- $\xi_{A_N}(\mathrm{v}_i \approx \mathrm{v}_j) = \{\vec{a} \in A^\omega : a_i = a_j\}$, for all $i, j \in N$,
- $\xi_{A_N}(\overline{x}) = x$, for every $x \in \mathrm{Rel}_N(A)$,
- $\xi_{A_N}(\neg\phi) = A^\omega - \xi_{A_N}(\phi)$, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N$,
- $\xi_{A_N}(\phi_1 \wedge \phi_2) = \xi_{A_N}(\phi_1) \cap \xi_{A_N}(\phi_2)$, for all $\phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(\mathrm{Rel}(A))_N$,
- $\xi_{A_N}(\exists_k\phi) = \{\vec{b} \in A^\omega : a_i = b_i \ \forall i \neq k \quad \text{and} \quad \vec{a} \in \xi_{A_{N\cup\{k\}}}(\phi)\}$.

It is shown in [19] that ξ_A is indeed an **HSet**-morphism and that $\mathbf{A}^*$ is a **T**-algebra.

It is not difficult to see that $\mathbf{A}^*$ is in $P(|\overrightarrow{\mathrm{Lf}_\omega}|)$. In fact, if $\mathbf{Rel}(A)$ is the largest finite dimensional subalgebra of the full ω-dimensional cylindric set algebra with universe $\mathrm{Rel}(A) = \mathcal{P}(A^\omega)$, then $P(\mathbf{Rel}(A)) = \mathbf{A}^*$. Thus, although the algebraic theory **T** in **HSet** used in [19] to algebraize the institution of first-order logic without terms "corresponds" [12] to a variety strictly larger than the variety of cylindric algebras, the subcategory **Q** of the category of all **T**-algebras, used as the category of algebras of the algebraic counterpart of the institution, corresponds to the full subcategory of the category of all cylindric algebras consisting of the largest finite dimensional subalgebras of the full ω-dimensional cylindric set algebras over all possible relational universes.

4 A Direction for Investigation The main logic that has been studied heavily in the literature and for which a categorical algebraization process has not yet been investigated is the full first-order logic with terms. Many institutions for this logic have been presented in the literature (see, e.g., [7]) and an attempt to provide an algebraic counterpart for that logic by extending the theory of cylindric algebras has been presented in [5]. It would be very interesting to investigate the possibility of a categorical algebraization of first-order logic, as expressed by a possibly modified version of the institutions presented in the literature, and then explore the connections between the algebraic theory used in the algebraization and the variety of algebras introduced in [5].

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