DENSITY OF THE SET OF ALL INFINITELY DIFFERENTIABLE FUNCTIONS WITH COMPACT SUPPORT IN WEIGHTED SOBOLEV SPACES

EIICHI NAKAI, NAOHITO TOMITA AND KÔZÔ YABUTA

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Abstract. Miller (1982) stated that, without a proof, if $1 < p < \infty$ and ω is in Muckenhoupt's A_p-class, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in weighted Sobolev spaces $L^p_k(\mathbb{R}^n, \omega(x)dx)$. In this paper, we show this property in the case $p = 1$ which is the critical case. We also give a proof in the case $1 < p < \infty$ for convenience. Generalizing the case $1 < p < \infty$, we can prove the density for Orlicz-Sobolev, Lorentz-Sobolev and Herz-Sobolev spaces with A_p-weights.

1. Introduction

Let $1 \le p < \infty$. For any non-negative integer k, the Sobolev space $L^p_k(\mathbb{R}^n)$ is defined as the space of functions f, with $f \in L^p(\mathbb{R}^n)$ and all $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ exist and $\dfrac{\partial^\alpha f}{\partial x^\alpha} \in L^p(\mathbb{R}^n)$ in the weak sense, whenever $|\alpha| \le k$. The Sobolev space $L^p_k(\mathbb{R}^n)$ is complete with the norm

$$\|f\|_{L^p_k} = \sum_{|\alpha|\le k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_p, \quad \left(\frac{\partial^0 f}{\partial x^0} = f \right),$$

where $\|\cdot\|_p$ is the usual norm of $L^p(\mathbb{R}^n)$. It is well known that $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, the set of all infinitely differentiable functions with compact support, is dense in $L^p_k(\mathbb{R}^n)$.

For a weight function ω, the weighted Sobolev space $L^p_k(\omega) = L^p_k(\mathbb{R}^n, \omega(x)dx)$ is defined by using $L^p(\omega) = L^p(\mathbb{R}^n, \omega(x)dx)$ instead of $L^p(\mathbb{R}^n)$. Miller [8] stated that, without a proof, if $1 < p < \infty$ and ω is in Muckenhoupt's A_p-class, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is also dense in weighted Sobolev spaces $L^p_k(\omega)$.

In this paper, we show this property in the case $p = 1$ which is the critical case (see [9, 6.6 in p.160]). We also give a proof in the case $1 < p < \infty$ for convenience. Generalizing the case $1 < p < \infty$, we can prove the density for Orlicz-Sobolev, Lorentz-Sobolev and Herz-Sobolev spaces with A_p-weights.

We give the definition of A_p in the next section. Our main results are the following:

Theorem 1.1. *Let $1 \le p < \infty$, $\omega \in A_p$ and k is a non-negative integer. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^p_k(\omega)$.*

In general, even if $f \in L^1(\omega)$, $f(\cdot - y)$ is not necessarily in $L^1(\omega)$. We show in the next section if $f \in L^1(\omega)$ with $\omega \in A_1$ then $f(\cdot - y)$ is in $L^1(\omega)$ a.e. y (see Remark 2.1).

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Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the set of all locally integrable functions on $\mathbb{R}^n$. The Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing x.

The case $1 < p < \infty$ in Theorem 1.1 can be generalized as follows. Let E be a subspace of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ equipped with a norm or quasi-norm $\|\cdot\|_E$. Let E_k be the space of all functions $f \in E$ such that $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ exist in the weak sense and $\dfrac{\partial^\alpha f}{\partial x^\alpha} \in E$ whenever $|\alpha| \leq k$. Then the space E_k is a norm or quasi-norm space with

$$\|f\|_{E_k} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_E, \quad \left(\frac{\partial^0 f}{\partial x^0} = f \right).$$

Theorem 1.2. *Let k be a non-negative integer and E have the following properties:*

1. *The characteristic functions of all balls in $\mathbb{R}^n$ are in E.*
2. *If $g \in E$ and $|f(x)| \leq |g(x)|$ a.e., then $f \in E$.*
3. *If $g \in E$, $|f_j(x)| \leq |g(x)|$ a.e. $(j = 1, 2, \cdots)$ and $f_j(x) \to 0$ $(j \to +\infty)$ a.e., then $f_j \to 0$ $(j \to +\infty)$ in E.*

If the operator M is bounded on E, then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in E_k.

In the next section we prove the theorems. In the third section we state applications of Theorem 1.2 to Orlicz-Sobolev, Lorentz-Sobolev and Herz-Sobolev spaces with A_p-weights.

2. Proof

First, we give the definition of Muckenhoupt's A_p-class, $1 \leq p < \infty$. A non-negative locally integrable function ω is said to belong to A_p, denoted $\omega \in A_p$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} < +\infty \quad (1 < p < \infty),$$

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\operatorname{ess\,sup}_{x \in Q} \omega(x)^{-1} \right) < +\infty \quad (p = 1),$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

In the case $1 < p < \infty$, the operator M is bounded on $L^p(\omega)$ if and only if $\omega \in A_p$. In the case $p = 1$, $M\omega(x) \leq C\omega(x)$ a.e. $x \in \mathbb{R}^n$ if and only if $\omega \in A_1$. See [3] for example.

For a function ψ on $\mathbb{R}^n$ and $t > 0$, we define

$$\psi_t(x) = \frac{1}{t^n} \psi\left(\frac{x}{t}\right).$$

To prove the theorems, we state two lemmas. For the proof of the first lemma, see [2, Proposition 2.7] or [9, p.63].

Lemma 2.1. *Let ψ be a function on $\mathbb{R}^n$ which is non-negative, radial, decreasing (as a function on $(0, \infty)$) and integrable. Then*

$$\sup_{t > 0} |(\psi_t * f)(x)| \leq \|\psi\|_1 \, Mf(x), \quad x \in \mathbb{R}^n.$$

The following is a key lemma to prove Theorem 1.1 in the case $p = 1$.

Lemma 2.2. *Let $1 \le p < \infty$ and ψ be as in Lemma 2.1. If $\omega \in A_p$, then there exists a constant $C > 0$ such that*

$$\|\psi_t * f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)} \quad \textit{for} \quad f \in L^p(\omega),$$

where C is independent of $t > 0$.

Proof. The case $p = 1$: Using $\psi(y) = \psi(-y)$, Lemma 2.1, $M\omega(x) \le C\omega(x)$ a.e. $x \in \mathbb{R}^n$ and Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\psi_t * f(x)|\, \omega(x)\, dx &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-y)|f(y)|\,\omega(x)\, dydx \\ &= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \psi_t(y-x)\omega(x)\, dx\right) dy \\ &\le \int_{\mathbb{R}^n} |f(y)|\, \|\psi\|_1 M\omega(y)\, dy \\ &\le C\|\psi\|_1 \int_{\mathbb{R}^n} |f(y)|\,\omega(y)\, dy.\end{aligned} \tag{2.1}$$

The case $1 < p < \infty$: Using Lemma 2.1 and the boundedness of M on $L^p(\omega)$, we have the conclusion. □

Remark 2.1. From (2.1) it follows that

$$\begin{aligned}\int_{R^n} \psi_t(y) \left(\int_{R^n} |f(x-y)|\,\omega(x)\, dx\right) dy &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \psi_t(y)|f(x-y)|\, dy\right) \omega(x)\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_t(x-y)|f(y)|\,\omega(x)\, dydx \\ &\le C\|\psi\|_1 \|f\|_{L^1(\omega)}.\end{aligned}$$

Let ψ be the Poisson kernel. Then $\psi(y) > 0$ for all $y \in \mathbb{R}^n$. Fubini's theorem shows that if $f \in L^1(\omega)$ with $\omega \in A_1$ then $f(\cdot - y) \in L^1(\omega)$ a.e. y.

Proof of Theorem 1.1. Let $L^p_{k,\mathrm{comp}}(\omega)$ be the set of all $f \in L^p_k(\omega)$ with compact support. Then $L^p_{k,\mathrm{comp}}(\omega)$ is dense in $L^p_k(\omega)$. Let $\psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ be a non-negative, radial and decreasing function with $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$. We show that, for all $f \in L^p_{k,\mathrm{comp}}(\omega)$,

$$\|\psi_t * f - f\|_{L^p_k(\omega)} \to 0 \quad \text{as} \quad t \to 0.$$

Since

$$\frac{\partial^\alpha(\psi_t * f)}{\partial x^\alpha} = \psi_t * \frac{\partial^\alpha f}{\partial x^\alpha},$$

it is enough to show that, for $f \in L^p_{\mathrm{comp}}(\omega)$,

$$\|\psi_t * f - f\|_{L^p(\omega)} \to 0 \quad \text{as} \quad t \to 0. \tag{2.2}$$

The case $p = 1$: Since $L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^1(\omega)$, for $\epsilon > 0$ we can take a function $g \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ such that $\|f - g\|_{L^1(\omega)} < \epsilon$. Using Lemma 2.2, we have that

$$\begin{aligned}&\|\psi_t * f - f\|_{L^1(\omega)} \\ &\le \|\psi_t * f - \psi_t * g\|_{L^1(\omega)} + \|\psi_t * g - g\|_{L^1(\omega)} + \|g - f\|_{L^1(\omega)} \\ &\le C\|f - g\|_{L^1(\omega)} + \|\psi_t * g - g\|_{L^1(\omega)} + \|g - f\|_{L^1(\omega)} \\ &\le (C+1)\epsilon + \|\psi_t * g - g\|_{L^1(\omega)}.\end{aligned}$$

We note that $\psi_t * g(x) \to g(x)$ a.e. x as $t \to 0$. From $g \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ it follows that $\|\psi_t * g\|_\infty \le \|g\|_\infty$ and that $\operatorname{supp} \psi_t * g$ is included in a certain ball for $0 < t < 1$. Therefore, by Lebesgue's convergence theorem, we have $\|\psi_t * g - g\|_{L^1(\omega)} \to 0$ as $t \to 0$. This shows (2.2).

The case $1 < p < \infty$: We note that $\psi_t * f(x) \to f(x)$ a.e. x as $t \to 0$. From Lemma 2.1 and the boundedness of M on $L^p(\omega)$ it follows that $|\psi_t * f(x)| \le \|\psi\|_1 Mf(x)$ a.e. x and $Mf \in L^p(\omega)$. Therefore, by Lebesgue's convergence theorem, we have (2.2). □

Proof of Theorem 1.2. From (1) and (2) it follows that $C^\infty_{\mathrm{comp}}(\mathbb{R}^n) \subset E_k$. Let $E_{k,\mathrm{comp}}$ be the set of all $f \in E_k$ with compact support. Then, using smooth cut-off functions and the property (3), we have that $E_{k,\mathrm{comp}}$ is dense in E_k. Since the operator M is bounded, we can use the same method as Theorem 1.1 in the case $1 < p < \infty$, and we have the conclusion. □

3. Applications of Theorem 1.2

Weighted Orlicz, Lorentz, Herz spaces, and their generalizations have the properties (1) and (2). Therefore, in the case that the property (3) holds and the operator M is bounded, we can apply Theorem 1.2.

For a weight function ω and a measurable set $\Omega \subset \mathbb{R}^n$, let $\omega(\Omega) = \int_\Omega \omega(x)\,dx$ and let χ_Ω be the characteristic function of Ω.

3.1. **Weighted Orlicz-Sobolev spaces.**

A function $\Phi : [0, +\infty] \to [0, +\infty]$ is called a Young function if Φ is convex, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = \Phi(+\infty) = +\infty$. Any Young function is increasing. A Young function Φ is said to satisfy Δ_2 condition, denoted $\Phi \in \Delta_2$, if $\Phi(2r) \le C\Phi(r)$ for some constant $C > 0$. If $\Phi \in \Delta_2$, then Φ satisfies

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty.$$

In this case Φ is continuous and bijective from $[0, +\infty)$ to itself. For a Young function Φ, the complementary function $\widetilde{\Phi}$ is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \ge 0\}, \quad r \ge 0.$$

For a Young function Φ and a weight function ω, let

$$L^\Phi(\omega) = L^\Phi(\mathbb{R}^n, \omega(x)dx) = \left\{ f : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\,\omega(x)\,dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{L^\Phi(\omega)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)\omega(x)\,dx \le 1 \right\}.$$

For a Young function $\Phi \in \Delta_2$, let

$$i(\Phi) = \lim_{\lambda\to 0+} \frac{\log h_\Phi(\lambda)}{\log\lambda} = \sup_{0<\lambda<1} \frac{\log h_\Phi(\lambda)}{\log\lambda}, \quad h_\Phi(\lambda) = \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}.$$

If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\omega) = L^p(\omega)$, $\|f\|_{L^\Phi(\omega)} = \|f\|_{L^p(\omega)}$ and $i(\Phi) = p$.

If $f_j \to 0$ in $L^\Phi(\omega)$ as $j \to +\infty$, then

$$\int_{\mathbb{R}^n} \Phi(|f_j(x)|)\,\omega(x)\,dx \to 0 \quad \text{as} \quad j \to +\infty.$$

If and only if $\Phi \in \Delta_2$, the converse is true. In this case $L^\Phi(\omega)$ has the property (3).

Let Φ and $\widetilde{\Phi}$ satisfy Δ_2 condition. Then the operator M is bounded on $L^\Phi(\omega)$ if and only if $\omega \in A_{i(\Phi)}$ (see [4, Theorem 2.1.1]).

The weighted Orlicz-Sobolev space $L^\Phi_k(\omega) = L^\Phi_k(\mathbb{R}^n, \omega(x)dx)$ is defined by using $L^\Phi(\omega) = L^\Phi(\mathbb{R}^n, \omega(x)dx)$ instead of $L^p(\mathbb{R}^n)$.

Corollary 3.1. *Let Φ and $\widetilde{\Phi}$ satisfy Δ_2 condition, $\omega \in A_{i(\Phi)}$, and k be a non-negative integer. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^\Phi_k(\omega)$.*

3.2. **Weighted Lorentz-Sobolev spaces.** Let ω be a weight function. For a measurable function f, the distribution function $\omega(f,s)$ and the rearrangement $f^*(t)$ with respect to the measure $\omega(x)dx$ are defined by

$$\omega(f,s) = \omega(\{x \in \mathbb{R}^n : |f(x)| > s\}) = \int_{\{x\in\mathbb{R}^n:|f(x)|>s\}} \omega(x)\,dx, \quad \text{for } s > 0,$$
$$f^*(t) = \inf\{s > 0 : \omega(f,s) \le t\}, \quad \text{for } t > 0.$$

The weighted Lorentz space $L^{(p,q)}(\omega) = L^{(p,q)}(\mathbb{R}^n, \omega(x)dx)$ is defined to be the set of all f such that $\|f\|_{L^{(p,q)}(\omega)} < \infty$, where

$$\|f\|_{L^{(p,q)}(\omega)} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty t^{(q/p)-1}(f^*(t))^q\,dt\right)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & 0 < p \le \infty,\ q = \infty. \end{cases}$$

If $0 < p = q \le \infty$, then $L^{(p,p)}(\omega) = L^p(\omega)$ and $\|f\|_{L^{(p,p)}(\omega)} = \|f\|_{L^p(\omega)}$.

If there exists a function g with $\lim_{t\to+\infty} g^*(t) = 0$ such that

$$\lim_{j\to+\infty} f_j(x) = 0 \quad \text{and} \quad |f_j(x)| \le |g(x)| \text{ a.e. } x \in \mathbb{R}^n,$$

then

$$\lim_{j\to+\infty} {f_j}^*(t) = 0 \quad \text{and} \quad {f_j}^*(t) \le g^*(t), \quad t > 0.$$

Actually, for all $s > 0$, we have

$$\chi_{\{y\in\mathbb{R}^n:|f_j(y)|>s\}}(x) \le \chi_{\{y\in\mathbb{R}^n:|g(y)|>s\}}(x) \text{ a.e. } x \in \mathbb{R}^n,$$
$$\text{and} \quad \omega(g,s) < +\infty, \text{ i.e. } \chi_{\{y\in\mathbb{R}^n:|g(y)|>s\}} \in L^1(\omega).$$

By Lebesgue's convergence theorem, we have $\omega(f_j, s) \to 0$ $(j \to +\infty)$ for all $s > 0$.

If $g \in L^{(p,q)}(\omega)$ with $0 < p < \infty$, then $\lim_{t\to+\infty} g^*(t) = 0$. Therefore, if $0 < p < \infty$ and $0 < q < \infty$, then $L^{(p,q)}(\omega)$ has the property (3).

Let $1 < p < \infty$, $1 \le q < \infty$, $\omega \in A_p$, Then the operator M is bounded on $L^{(p,q)}(\omega)$ (see [1, Theorems 3 and 4]).

The weighted Lorentz-Sobolev space $L^{(p,q)}_k(\omega) = L^{(p,q)}_k(\mathbb{R}^n, \omega(x)dx)$ is defined by using $L^{(p,q)}(\omega) = L^{(p,q)}(\mathbb{R}^n, \omega(x)dx)$ instead of $L^p(\mathbb{R}^n)$.

Corollary 3.2. *Let $1 < p < \infty$, $1 \le q < \infty$, $\omega \in A_p$, and k is a non-negative integer. Then $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is dense in $L^{(p,q)}_k(\omega)$.*

3.3. **Weighted Herz-Sobolev spaces.** Let $B_k = B(0, 2^k)$ for $k \in \mathbb{Z}$ and $R_k = B_k \setminus B_{k-1}$. Let $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, ω_1 and ω_2 be weight functions.

The homogeneous weighted Herz space $\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$ is defined to be the set of all f such that $\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} < \infty$, where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} = \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_{R_k}\|^p_{L^q(\omega_2)}\right)^{1/p},$$

with the usual modifications when $p = \infty$ and/or $q = \infty$.

The non-homogeneous weighted Herz space $K^{\alpha,p}_q(\omega_1,\omega_2)$ is defined to be the set of all f such that $\|f\|_{K^{\alpha,p}_q(\omega_1,\omega_2)} < \infty$, where

$$\|f\|_{K^{\alpha,p}_q(\omega_1,\omega_2)} = \left(\omega_1(B_0)^{\alpha p/n}\|f\chi_{B_0}\|^p_{L^q(\omega_2)} + \sum_{k=1}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_{R_k}\|^p_{L^q(\omega_2)}\right)^{1/p},$$

with the usual modifications when $p = \infty$ and/or $q = \infty$.

If $\alpha = 0$ and $0 < p = q \le \infty$, then $\dot{K}_p^{\alpha,p}(\omega_1, \omega_2) = K_p^{\alpha,p}(\omega_1, \omega_2) = L^p(\omega_2)$ and $\|f\|_{\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)} = \|f\|_{K_p^{\alpha,p}(\omega_1,\omega_2)} = \|f\|_{L^p(\omega_2)}$.

If $0 < p, q < \infty$, then $\dot{K}_p^{\alpha,q}(\omega_1, \omega_2)$ and $K_p^{\alpha,q}(\omega_1, \omega_2)$ have the property (3).

If α, p, q, ω_1 and ω_2 satisfy the assumption in the next corollary, then the operator M is bounded on $\dot{K}_p^{\alpha,q}(\omega_1, \omega_2)$ and on $K_p^{\alpha,q}(\omega_1, \omega_2)$ (see [5, Theorem 1]). Actually, Professor Lu pointed out

$$\begin{aligned} Mf(x) &\le C \sup_{r>0} \frac{1}{r^n} \int_{|y-x|<r} |f(x)|\, dy \\ &\le C \sup_{r>0} \int_{|y-x|<r} \frac{|f(x)|}{|y-x|^n}\, dy \\ &\le C \int_{\mathbb{R}^n} \frac{|f(x)|}{|y-x|^n}\, dy. \end{aligned}$$

The weighted Herz-Sobolev space $\dot{K}_{q,k}^{\alpha,p}(\omega_1, \omega_2)$ and $K_{q,k}^{\alpha,p}(\omega_1, \omega_2)$ is defined by using $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ and $K_q^{\alpha,p}(\omega_1, \omega_2)$, respectively, instead of $L^p(\mathbb{R}^n)$.

Corollary 3.3. *Let $\omega_1 \in A_{q_{\omega_1}}$, $\omega_2 \in A_{q_{\omega_2}}$, $0 < p < \infty$, $1 < q < \infty$ and k is a non-negative integer, where ω_1 and ω_2 satisfy either of the following,*

(i) *$\omega_1 = \omega_2$, $1 \le q_{\omega_1} \le q$ and $-nq_{\omega_1}/q < \alpha q_{\omega_1} < n(1 - q_{\omega_1}/q)$,*
(ii) *$1 \le q_{\omega_1} < \infty$, $1 \le q_{\omega_2} \le q$ and $0 < \alpha q_{\omega_1} < n(1 - q_{\omega_2}/q)$.*

Then $C_{\mathrm{comp}}^\infty(\mathbb{R}^n)$ is dense in $\dot{K}_{q,k}^{\alpha,p}(\omega_1, \omega_2)$ and in $K_{q,k}^{\alpha,p}(\omega_1, \omega_2)$.

In the case $\omega_1(x) = \omega_2(x) \equiv 1$, we denote $\dot{K}_{q,k}^{\alpha,p}(\omega_1, \omega_2)$ and $K_{q,k}^{\alpha,p}(\omega_1, \omega_2)$ by $\dot{K}_{q,k}^{\alpha,p}(\mathbb{R}^n)$ and $K_{q,k}^{\alpha,p}(\mathbb{R}^n)$, respectively. It is known that $C_{\mathrm{comp}}^\infty(\mathbb{R}^n)$ is dense in $\dot{K}_{q,k}^{\alpha,p}(\mathbb{R}^n)$ and in $K_{q,k}^{\alpha,p}(\mathbb{R}^n)$ if $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1 - 1/q)$ and k is a non-negative integer ([6, Proposition 2.1]).

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Eiichi Nakai, Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan

E-mail address: enakai@cc.osaka-kyoiku.ac.jp

Naohito Tomita, Department of Mathematics, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka 560-0043, Japan
E-mail address: tomita@gaia.math.wani.osaka-u.ac.jp

Kôzô Yabuta, School of Science and Technology, Kwansei Gakuin University, Gakuen 2-1, Sanda 669-1337, Japan
E-mail address: yabuta@ksc.kwansei.ac.jp