ON PROPERTIES OF SOLUTIONS ANNIHILATED BY A COMPLEX VECTOR FIELD IN $\mathbb{R}^2$

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Abstract. Let X be a nowhere-zero C^∞ complex vector field defined near the origin in $\mathbb{R}^2$. We may suppose that X has the form of $\frac{\partial}{\partial t} + ir(t,x)\frac{\partial}{\partial x}$, where $r(t,x)$ is a real-valued C^∞ function. Up to now the investigations on local integrability for vector fields X satisfying $r(0,x) = 0$ have been focused. This paper treats the vector field X of the form of

$$\frac{\partial}{\partial t} + i\big(t^d + a(x)\big)\frac{\partial}{\partial x},$$

where d is a positive integer and $a(x)$ a real-valued C^∞ function satisfying $a(0) = 0$. Under certain assumptions on $a(x)$, the following properties are shown:

Property A. *Every C^1 solution u of the equation $Xu = 0$ in a neighborhood of the origin which satisfies that $u(0,x)$ is constant is identically constant.*

Property B. *Every C^2 solution u of the equation $Xu = 0$ in a neighborhood of the origin which satisfies $u(t,x) - u(-t,x) = o(t^2)(t \to 0)$ is identically constant.*

1. Introduction

Let X be a nowhere-zero C^∞ complex vector field defined near the origin in $\mathbb{R}^2$. It is said that X is locally integrable at the origin if there exist a neighborhood ω of the origin and function u satisfying $Xu = 0$ in ω such that $du \neq 0$. We may suppose that X has the form of $\frac{\partial}{\partial t} + i\big(t^d + a(x)\big)\frac{\partial}{\partial x}$, where $r(t,x)$ is a real-valued C^∞ function. X is locally integrable at the origin if $r(t,x) \geqq 0$ in a neighborhood of the origin. There are several studies on local integrability for non-solvable vector fields X([3],[4],[5],[7],[8],[9],[10], $\cdots$).

On the other-hand, Nirenberg [6](see also [2]) gave an example of X of the form of

$$\frac{\partial}{\partial t} + it\big(1 + t\rho(t^2,x)\big)\frac{\partial}{\partial x}$$

such that the $Xu = 0$ admits only constant solutions in any neighborhood of the origin, where $\rho(t,x)$ is a real-valued C^∞ function satisfying some conditions. (Incidentally, L.Hölmander[1] gave an example of X (satisfying $r(0,x) = 0$) such that the $Xu = 0$ admits a C^∞ solution in a neighborhood ω_0 of the origin which vanishes for $t \leqq 0$ with the $\operatorname{supp} u = \{(t,x); t \geqq 0\} \cap \omega_0$.)

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Up to now the investigations on local integrability for vector fields X satisfying $r(0,x)=0$ have been focused. In this paper we present an operator X of the form of

$$\frac{\partial}{\partial t}+i\big(t^d+a(x)\big)\frac{\partial}{\partial x}$$

which has the following properties:

Property A. *Every C^1 solution u of the equation $Xu=0$ in a neighborhood of the origin which satisfies that $u(0,x)$ is constant is identically constant.*

Property B. *Every C^2 solution u of the equation $Xu=0$ in a neighborhood of the origin which satisfies $u(t,x)-u(-t,x)=o(t^2)(t\to 0)$ is identically constant.*

2. Theorems

Let c_n and d_n be real constants satisfying

$$0<d_{n+1}<c_n<d_n<1(n=1,2,\cdots),\ \lim_{n\to\infty} d_n=0,$$

or

$$-1<c_n<d_n<c_{n+1}<0(n=1,2,\cdots),\ \lim_{n\to\infty} c_n=0.$$

We assume:
(a.1) $a(x)\in C^\infty((-1,1))$.
(a.2) $a(x)\equiv 0$ in $[c_n,d_n](n=1,2,\cdots)$ and $a(x)>0$ in $(-1,1)\setminus\cup_{n=1}^{\infty}[c_n,d_n]$.
(a.3) d is a positive integer.

Then we obtain the following:

Theorem A. *Let X be $\frac{\partial}{\partial t}+i\big(t^d+a(x)\big)\frac{\partial}{\partial x}$. Then every C^1 solution u of the equation $Xu=0$ in a neighborhood of the origin which satisfies that $u(0,x)$ is constant is identically constant.*

Theorem B. *Let X be $\frac{\partial}{\partial t}+i\big(t^d+a(x)\big)\frac{\partial}{\partial x}$. Then every C^2 solution u of the equation $Xu=0$ in a neighborhood of the origin which satisfies $u(t,x)-u(-t,x)=o(t^2)(t\to 0)$ is identically constant.*

Remark 1. Whatever a positive integer d, the vector field $X\equiv\frac{\partial}{\partial t}+it^d\frac{\partial}{\partial x}$ has the property such as stated in Theorem A but does not have the one such as stated in Theorem B:
$u\equiv\frac{t^{d+1}}{d+1}+ix$ is a non-constant solution of $Xu=0$ which satisfies $u(t,x)-u(-t,x)=o(t^2)(t\to 0)$.

Remark 2. Whatever an even number d, we know that the equation $Xu=\frac{\partial u}{\partial t}+i\big(t^d+a(x)\big)\frac{\partial u}{\partial x}=0$ has a smooth solution u in a neighborhood of the origin such that $u_x\neq 0$. So, there exists a non-constant solution u annihilated by X which does not satisfy $u(t,x)-u(-t,x)=o(t^2)(t\to 0)$.

3. Proof

Proof of Theorem A. Suppose the contrary. Then we may suppose that $Xu = 0$ has a C^1 solution u in a neighborhood of the origin ω such that $\frac{\partial u}{\partial x} \not\equiv 0$. Noting that the operator $X = \frac{\partial}{\partial t} + i\big(t^d + a(x)\big)\frac{\partial}{\partial x}$ is elliptic in $\{(t,x); t^d + a(x) \neq 0\}$, we find that $u \in C^\infty(\omega \cap \{(t,x); t^d + a(x) \neq 0\})$.

Differentiating the $Xu = 0$ with respect to x and setting $v = u_x$, we obtain

$$\frac{\partial v}{\partial t} + i\big(t^d + a(x)\big)\frac{\partial v}{\partial x} + ia'(x)v = 0 \quad in \quad \omega \cap \{(t,x); t^d + a(x) \neq 0\}.$$

Taking a sufficiently large integer n such that $[c_n, d_n] \subset \omega \cap \{(t,x) : t = 0\}$, we have

$$\Big(\frac{\partial}{\partial t} + it^d\frac{\partial}{\partial x}\Big)v = 0$$

in $\omega \cap (-\infty,\infty) \times [c_n, d_n] \cap \{(t,x); t \neq 0\}$.

Since $v(0,x) = u_x(0,x) = 0$ in $[c_n, d_n]$, at first, we find that v must vanish identically in $\omega \cap (-\infty,\infty) \times [c_n, d_n]$.

On the other hand, we may suppose that there exists a point $\mathrm{P}_0 \in \omega \cap \{(t,x); t^d + a(x) \neq 0\}$ such that $v\ (\mathrm{P}_0) \neq 0$. We take a simply connected domain $D \subset \omega$ with a smooth rectifiable boundary such that D does not intersect with $\{(t,x); t^d + a(x) \neq 0\}$, $\mathrm{P}_0 \in D$, and $D \cap (-\infty,\infty) \times (c_n, d_n) \neq \emptyset$.

Since X is elliptic in D, we find that there exists a smooth function Z such that $d\,Z \neq 0$ satisfying $XZ = 0$ in D. Then $X = X\overline{Z}\,\frac{\partial}{\partial \overline{Z}}$. We also find that there exists a smooth solution w satisfying $\frac{\partial w}{\partial \overline{Z}} = \frac{ia'(x)}{X\overline{Z}}$ in D. Thus we see that $\frac{\partial\left(v \exp w\right)}{\partial \overline{Z}} = 0$ holds in D. Therefore, from $v = 0$ in $\omega \cap (-\infty,\infty) \times [c_n, d_n]$, we can conclude that v vanishes identically in D, which contradicts $v(\mathrm{P}_0) \neq 0$.

Proof of Theorem B. Suppose the contrary. Then we may suppose that the $Xu = 0$ has a C^2 solution u in a neighborhood of the origin ω such that $\frac{\partial u}{\partial x} \not\equiv 0$. Differentiating the $Xu = 0$ with respect to x and setting $v = u_x$, we have

$$\frac{\partial v}{\partial t} + i\big(t^d + a(x)\big)\frac{\partial v}{\partial x} + ia'(x)v = 0 \quad in \quad \omega. \tag{1}$$

By taking the odd part of equation $Xu = 0$ with respect to t,

$$\Big(\frac{\partial}{\partial t} + it^d\frac{\partial}{\partial x}\Big)u^o = -ia(x)u^e_x, \tag{2}$$

where u^o denotes the odd part of u with respect to t and u^e the even one. Taking a sufficiently large integer N such that $[c_n, d_n] \subset \omega \cap \{(t,x) : t = 0\}$ for every $n > N$, we see the following

Lemma 1.

$$u^e_x(0,x) \not\equiv 0 \quad in \quad [c_n, d_n]$$

for every $n > N$.

Proof. Suppose that exists a positive integer $m > N$ such that

$$u^e_x(0,x) \equiv 0 \quad in \quad [c_m, d_m].$$

From (1), we have

$$\Big(\frac{\partial}{\partial t}+it^d\frac{\partial}{\partial x}\Big)v=0$$

in $\omega\cap(-\infty,\infty)\times[c_m,d_m]$.

Since $v(0,x)=u_x(0,x)=u_x^e(0,x)=0$ in $[c_m,d_m]$, we find that v must vanish identically in $\omega\cap(-\infty,\infty)\times[c_m,d_m]$ and hence we can conclude that v vanishes identically in ω, by making use of the same method such as used in Theorem A. This yields a contradiction.

From now on we take n such that $n>N$ and fix it. By Lemma 1, there exist real constants $a'_n, b'_n (a'_n<0<b'_n)$, $c'_n, d'_n (c_n \leqq c'_n < d'_n \leqq d_n)$ such that $\Re u_x^e(t,x)\neq 0$ or $\Im u_x^e(t,x)\neq 0$ holds in $[a'_n,b'_n]\times[c'_n,d'_n]$. We arbitrarily take positive constants $\varepsilon_1,\varepsilon_2$ such that $\varepsilon_1<\varepsilon_2<b'_n$. We have the following

Lemma 2.

$$-i\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)u_x^e\,dtdx=$$
$$\int_{\varepsilon_1}^{\varepsilon_2}-it^d\Big(u^o(t,d'_n)-u^o(t,c'_n)\Big)\,dt+$$
$$\int_{c'_n}^{d'_n}\Big(u^o(\varepsilon_2,x)-u^o(\varepsilon_1,x)\Big)\,dx.$$

Proof. Setting $v(x,y)=x-\frac{it^{d+1}}{d+1}$ and making use of (2), we have

$$-i\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)u_x^e\,dtdx=$$
$$-i\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)u_x^e v_x\,dtdx=$$
$$\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]}\Big(u_t^o+it^du_x^o\ Bigr)v_x\,dtdx=$$
$$\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]}\Big(u_t^ov_x-u_x^ov_t\Big)\,dtdx=$$
$$\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} d\Big(u^o(t,x)dv(t,x)\Big)\ =$$
$$\oint_{\partial[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} u^o(t,x)dv(t,x)=$$
$$\oint_{\partial[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} u^o(t,x)v_t(t,x)dt+u^o(t,x)v_x(t,x)dx=$$
$$\int_{\varepsilon_1}^{\varepsilon_2}-it^d\Big(u^o(t,d'_n)-u^o(t,c'_n)\Big)\,dt+\int_{c'_n}^{d'_n}\Big(u^o(\varepsilon_2,x)-u^o(\varepsilon_1,x)\Big)\,dx.$$

By this Lemma 3 we obtain the following

Lemma 4. *There exist positive constants C_1,C_2,C_3 and C_4 which are independent of ε_1 and ε_2 such that*

$$C_1\int_{c'_n}^{d'_n} a(x)\,dx\leqq C_2\frac{\varepsilon_2^{d+3}-\varepsilon_1^{d+3}}{\varepsilon_2-\varepsilon_1}+C_3(\varepsilon_1+\varepsilon_2)+C_4\varepsilon_1^2.$$

Proof. We see

$$\Big|\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)u^e_x\,dtdx\Big| \geqq$$

$$\frac{\Big|\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)\Re u^e_x dtdx\Big| + \Big|\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)\Im u^e_x dtdx\Big|}{\sqrt{2}}.$$

Now $\Re u^e_x(t,x) \neq 0$ or $\Im u^e_x(t,x) \neq 0$ in $[a'_n, b'_n] \times [c'_n, d'_n]$. Hence, when $\Re u^e_x \neq 0$, we see

$$\Big|\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)\Re u^e_x dtdx\Big| = \iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)\big|\Re u^e_x\big|dtdx \geqq$$

$$\min_{[a'_n,b'_n]\times[c'_n,d'_n]} \big|\Re u^e_x\big| \iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)dtdx =$$

$$\min_{[a'_n,b'_n]\times[c'_n,d'_n]} \big|\Re u^e_x\big|(\varepsilon_2-\varepsilon_1)\int_{c'_n}^{d'_n} a(x)dx,$$

and when $\Im u^e_x \neq 0$, in the similar way we have

$$\Big|\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)\Im u^e_x dtdx\Big| \geqq$$

$$\min_{[a'_n,b'_n]\times[c'_n,d'_n]} \big|\Im u^e_x\big|(\varepsilon_2-\varepsilon_1)\int_{c'_n}^{d'_n} a(x)dx.$$

Therefore we find that there exists a positive constant C_1 which is independent of ε_2 and ε_1 such that

$$\Big|\iint_{[\varepsilon_1,\varepsilon_2]\times[c'_n,d'_n]} a(x)u^e_x\,dtdx\Big| \geqq C_1(\varepsilon_2-\varepsilon_1)\int_{c'_n}^{d'_n} a(x)dx.$$

Now, by the assumption that $u \in C^2$ and $u^o = o(t^2)$, we see that the function $\frac{u^o(t,x)}{t^2} \in C^0(\omega)$. Hence we find that there exist positive constants $C_i (i = 2, 3, 4)$ which are independent of ε_2 and ε_1 such that

$$\Big|\int_{\varepsilon_1}^{\varepsilon_2} -it^d\Big(u^o(t,d'_n) - u^o(t,c'_n)\Big)\,dt\Big| \leqq C_2\int_{\varepsilon_1}^{\varepsilon_2}(d+3)t^{d+2}\,dt = C_2(\varepsilon_2^{d+3} - \varepsilon_1^{d+3})$$

and

$$\Big|\int_{c'_n}^{d'_n}\Big(u^o(\varepsilon_2,x) - u^o(\varepsilon_1,x)\Big)\,dx\Big| \leqq C_3(\varepsilon_2^2-\varepsilon_1^2) + C_4\varepsilon_1^2(\varepsilon_2-\varepsilon_1).$$

Lemma 4 is thus obtained, by applying Lemma 3.

Finally, letting $\varepsilon_2 \to 0$ in Lemma 4, we get the the contradiction that

$$0 < \int_{c'_n}^{d'_n} a(x)\,dx = 0.$$

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