FUZZIFICATIONS OF PSEUDO-BCI IDEALS

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Abstract. The fuzzification of pseudo-BCI ideals is considered, and some of their properties are investigated. Characterizations of pseudo-BCI ideals are provided. Also the homomorphic image and preimage of pseudo-BCI ideals are discussed.

1. Introduction

Georgescu and Iorgulescu [1] introduced the notion of a pseudo-BCK algebra as an extended notion of BCK-algebras. In [4], Jun, one of the present authors, gave a characterization of pseudo-BCK algebra, and provided conditions for a pseudo-BCK algebra to be \(\land\)-semi-lattice ordered (resp. \(\cap\)-semi-lattice ordered). Jun et al. [7] introduced the notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra, and then they investigated some of their properties. In [2], Dudek and Jun introduced the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigated some properties. Jun et al. [5] introduced the concepts of pseudo-atoms, pseudo-BCI ideals and pseudo-BCI homomorphisms in pseudo-BCI algebras. They displayed characterizations of a pseudo-BCI ideal, and provided conditions for a subset to be a pseudo-BCI ideal. They also introduced the notion of a \(\diamond\)-medial pseudo-BCI algebra, and gave its characterization. In this paper, we consider the fuzzification of pseudo-BCI ideals, and investigate some of their properties.

We give characterizations of pseudo-BCI ideals. We also discuss the homomorphic image and preimage of pseudo-BCI ideals.

2. Preliminaries

Definition 2.1. [2] A pseudo-BCI algebra is a structure \(X = (X, \preceq, *, \diamond, 0)\), where "\(\preceq\)" is a binary relation on a set \(X\), "\(*\)" and "\(\diamond\)" are binary operations on \(X\) and "\(0\)" is an element of \(X\), verifying the axioms: for all \(x, y, z \in X\),

(a1) \(x * y \cup (x * z) \preceq x * y, (x \diamond y) * (x \diamond z) \preceq z \diamond y,\)
(a2) \(x * (x \diamond y) \preceq y, x \diamond (x * y) \preceq y,\)
(a3) \(x \preceq x,\)
(a4) \(x \preceq y, y \preceq x \implies x = y,\)
(a5) \(x \preceq y \iff x * y = 0 \iff x \diamond y = 0.\)

Note that every pseudo-BCI algebra satisfying \(x * y = x \diamond y\) for all \(x, y \in X\) is a BCI-algebra. Every pseudo-BCK algebra is a pseudo-BCI algebra.

Proposition 2.2. [2] In a pseudo-BCI algebra \(X\) the following holds:

(p1) \(x \preceq 0 \implies x = 0.\)
(p2) \(x \preceq y \implies z * y \preceq z * x, z \diamond y \preceq z \diamond x.\)
(p3) \(x \preceq y, y \preceq z \implies x \preceq z.\)

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Proof. The proof is straightforward.

Theorem 3.5. Let \( \mu \) be a fuzzy pseudo-\( \text{BCI} \) ideal of \( X \) such that

\[
\begin{align*}
(p4) \quad (x \ast y) \ast z &= (x \ast z) \ast y, \\
(p5) \quad x \ast y \preceq z &\iff x \ast z \preceq y, \\
(p6) \quad (x \ast y) \ast (z \ast y) &\preceq x \ast (y \ast z), \\
(p7) \quad x \preceq y &\implies x \ast z \preceq y \ast z, \\
(p8) \quad x \ast 0 &= x = x \ast 0, \\
(p9) \quad x \ast (x \ast (x \circ y)) &= x \ast y \quad \text{and} \quad x \circ (x \ast (y \circ y)) = x \circ y.
\end{align*}
\]

3. Fuzzy Pseudo-\( \text{BCI} \) ideals

In what follows let \( \mathfrak{X} \) denote a pseudo-\( \text{BCI} \) algebra unless otherwise specified. For any nonempty subset \( J \) of \( X \) and any element \( y \) of \( X \), we denote

\[
*(y, J) := \{ x \in X \mid x \ast y \in J \} \quad \text{and} \quad \circ (y, J) := \{ x \in X \mid x \circ y \in J \}.
\]

Note that \( *(y, J) \cap \circ (y, J) = \{ x \in X \mid x \ast y \in J, x \circ y \in J \} \).

Definition 3.1. [5] A nonempty subset \( J \) of \( \mathfrak{X} \) is called a pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \) if it satisfies

\[
\begin{align*}
(\text{i}) & \quad 0 \in J, \\
(\text{ii}) & \quad \forall y \in J, *(y, J) \subseteq J \quad \text{and} \quad \circ (y, J) \subseteq J.
\end{align*}
\]

Note that if \( \mathfrak{X} \) is a pseudo-\( \text{BCI} \) algebra satisfying \( x \ast y = x \circ y \) for all \( x, y \in X \), then the notion of a pseudo-\( \text{BCI} \) ideal and a \( \text{BCI} \)-ideal coincide.

Definition 3.2. A fuzzy set \( \mu : \mathfrak{X} \to [0, 1] \) is called a fuzzy pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \) if for every \( t \in \text{Im}(\mu) \), \( U(\mu; t) \) is a pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \).

Theorem 3.3. A fuzzy set \( \mu : \mathfrak{X} \to [0, 1] \) is a fuzzy pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \) if and only if it satisfies:

\[
\begin{align*}
(\text{i}) & \quad \mu(0) \geq \mu(x), \forall x \in X, \\
(\text{ii}) & \quad \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}, \forall x, y \in X, \\
(\text{iii}) & \quad \mu(a) \geq \min\{\mu(a \circ b), \mu(b)\}, \forall a, b \in X.
\end{align*}
\]

Proof. Assume that \( \mu \) is a fuzzy pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \) and suppose that there exists \( x_0 \in X \) such that \( \mu(0) < \mu(x_0) \). Taking \( t_0 := \frac{1}{2}(\mu(0) + \mu(x_0)) \), we get \( \mu(0) < t_0 < \mu(x_0) \). Hence \( 0 \notin U(\mu; t_0) \), which is a contradiction. Therefore \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Assume that (ii) is false. Then

\[
\mu(x_1) < \min\{\mu(x_1 \ast y_1), \mu(y_1)\}
\]

for some \( x_1, y_1 \in X \). Putting

\[
t_1 := \frac{1}{2}(\mu(x_1) + \min\{\mu(x_1 \ast y_1), \mu(y_1)\}),
\]

we have \( \mu(x_1) < t_1 < \min\{\mu(x_1 \ast y_1), \mu(y_1)\} \). It follows that \( x_1 \ast y_1 \in U(\mu; t_1) \) and \( y_1 \in U(\mu; t_1) \), but \( x_1 \notin U(\mu; t_1) \). This is a contradiction. Hence (ii) is true. Similarly we have (iii).

Corollary 3.4. Let \( \mu \) be a fuzzy pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \). If \( x \preceq y \) in \( \mathfrak{X} \), then \( \mu(x) \geq \mu(y) \).

Proof. The proof is straightforward.

We give conditions for a fuzzy set in \( \mathfrak{X} \) to be a fuzzy pseudo-\( \text{BCI} \) ideal of \( \mathfrak{X} \).

Theorem 3.5. Let \( \mu \) be a fuzzy set in \( \mathfrak{X} \) such that

\[
\begin{align*}
(\text{i}) & \quad \mu(0) \geq \mu(x), \forall x \in X, \\
(\text{ii}) & \quad \mu(x \circ y) \geq \min\{\mu((x \ast y) \circ y) \ast z), \mu(z)\}, \forall x, y, z \in X, \\
(\text{iii}) & \quad \mu(a \ast b) \geq \min\{\mu((a \circ b) \ast b) \circ c), \mu(c)\}, \forall a, b, c \in X.
\end{align*}
\]
Theorem 3.7. Let $\mu$ be a fuzzy set satisfying the following properties

(i) $\forall x, y, z \in X, x \ast y \leq z \Rightarrow \mu(x) \geq \min\{\mu(x),\mu(y)\}$. 
(ii) $\forall a, b, c \in X, a \circ b \leq a \Rightarrow \mu(c) \geq \min\{\mu(a),\mu(b)\}$. 

Proof. (i) Let $x, y, z \in X$ be such that $z \ast y \leq x$. Then $\mu(z \ast y) \geq \mu(x)$ by Corollary 3.4, and so

$$\mu(z) \geq \min\{\mu(z \ast y),\mu(y)\} \geq \min\{\mu(x),\mu(y)\}$$

by Theorem 3.3(i). Now let $a, b, c \in X$ be such that $c \circ b \leq a$. Using Corollary 3.4, we get $\mu(c \circ b) \geq \mu(a)$. It follows from Theorem 3.3(ii) that

$$\mu(c) \geq \min\{\mu(c \circ b),\mu(b)\} \geq \min\{\mu(a),\mu(b)\}.$$ 

This completes the proof.

The following is obvious from Theorem 3.3.

Theorem 3.8. Let $\mu : X \to [0,1]$ be a fuzzy set satisfying the following properties

(i) $\forall x \in X, x \ast y \leq z \Rightarrow \mu(x) \geq \min\{\mu(y),\mu(z)\}$. 
(ii) $\forall a, b \in X, a \circ b \leq a \Rightarrow \mu(a) \geq \min\{\mu(b),\mu(c)\}$. 

Then $\mu$ is a fuzzy pseudo-$BCI$ ideal of $X$.

Proposition 3.9. A fuzzy set $\mu : X \to [0,1]$ is called a fuzzy pseudo-$BCI$ subalgebra of $X$ if and only if the nonempty level set $U(\mu; t)$ is a pseudo-$BCI$ subalgebra of $X$ where $t \in \text{Im}(\mu)$. 

Proof. Taking $y = 0$ in (ii) and using (p8), we know that

$$\mu(x) \geq \min\{\mu(x \ast z),\mu(z)\}$$

for all $x, z \in X$. If we take $b = 0$ in (iii) and use the condition (p8), then

$$\mu(a) \geq \min\{\mu(a \circ c),\mu(c)\}$$

for all $a, c \in X$. Hence, by Theorem 3.3, we have that $\mu$ is a fuzzy pseudo-ideal of $X$.

Theorem 3.10. If $\mu : X \to [0,1]$ is a fuzzy pseudo-$BCI$ ideal of $X$, then

(i) $\forall x, y, z \in X, x \ast y \leq z \Rightarrow \mu(x) \geq \min\{\mu(y),\mu(z)\}$. 
(ii) $\forall a, b, c \in X, a \circ b \leq a \Rightarrow \mu(c) \geq \min\{\mu(a),\mu(b)\}$. 

Proof. (i) Let $x, y, z \in X$ be such that $z \ast y \leq x$. Then $\mu(z \ast y) \geq \mu(x)$ by Corollary 3.4, and so

$$\mu(z) \geq \min\{\mu(z \ast y),\mu(y)\} \geq \min\{\mu(x),\mu(y)\}$$

by Theorem 3.3(i). Now let $a, b, c \in X$ be such that $c \circ b \leq a$. Using Corollary 3.4, we get $\mu(c \circ b) \geq \mu(a)$. It follows from Theorem 3.3(ii) that

$$\mu(c) \geq \min\{\mu(c \circ b),\mu(b)\} \geq \min\{\mu(a),\mu(b)\}.$$ 

This completes the proof.
Theorem 3.11. For a fuzzy set $\mu$ in $X$, let $\mu^\vee$ be a fuzzy set in $X$ defined by
\[
\mu^\vee(x) := \sup \{t \in [0, 1] \mid x \in \langle U(\mu; t) \rangle \}, \forall x \in X.
\]
Then $\mu^\vee$ is the least fuzzy pseudo-BCI ideal of $X$ that contains $\mu$, where $\langle U(\mu; t) \rangle$ means the least pseudo-BCI ideal of $X$ containing $U(\mu; t)$.

Proof. At first we shall show
\[
U(\mu^\vee; t) = \bigcap_{\epsilon > 0} (U(\mu; t - \epsilon)).
\]

Let $x \in U(\mu^\vee; t)$. Since $\mu^\vee(x) \geq t$, for every $\epsilon > 0$ there exists $t^*$ such that
\[
x \in U(\mu^\vee; t^*) \quad \text{and} \quad t^* > t - \epsilon.
\]

We have $U(\mu^\vee; t^*) \subseteq U(\mu; t - \epsilon)$ by $t^* > t - \epsilon$. It follows that for every $\epsilon > 0$
\[
x \in U(\mu; t - \epsilon) \subseteq \langle U(\mu; t - \epsilon) \rangle
\]
and hence that
\[
x \in \bigcap_{\epsilon > 0} (U(\mu; t - \epsilon)).
\]

Conversely, for every $x \in X$, we have
\[
x \in \bigcap_{\epsilon > 0} (U(\mu; t - \epsilon)) \implies x \in \langle U(\mu; t - \epsilon) \rangle \text{ for every } \epsilon > 0
\]
\[
\implies t - \epsilon \in \{t \in [0, 1] \mid x \in \langle U(\mu; t) \rangle \} \text{ for every } \epsilon > 0
\]
\[
\implies t - \epsilon \leq \sup \{t \in [0, 1] \mid x \in \langle U(\mu; t) \rangle \} = \mu^\vee(x) \text{ for every } \epsilon > 0
\]
\[
\implies t \leq \mu^\vee(x)
\]
\[
\implies x \in U(\mu^\vee; t)
\]

Hence we have $U(\mu^\vee; t) = \bigcap_{\epsilon > 0} (U(\mu; t - \epsilon))$.

It is obvious that
\[
U(\mu; t) = \bigcap_{\epsilon > 0} U(\mu; t - \epsilon).
\]

Since $\langle U(\mu; t - \epsilon) \rangle$ is a pseudo-BCI ideal and contains $U(\mu; t)$, we can conclude that $U(\mu^\vee; t) = \bigcap_{\epsilon > 0} (U(\mu; t - \epsilon))$ is the least pseudo-BCI ideal containing $U(\mu; t)$. It follows from transfer principle ([6]) that $\mu^\vee$ is the least fuzzy pseudo-BCI ideal of $X$ containing $\mu$. \qed

Definition 3.12. [5] Let $X$ and $Y$ be BCI algebras. A mapping $f : X \to Y$ is called a pseudo-BCI homomorphism if $f(x * y) = f(x) * f(y)$ and $f(x \circ y) = f(x) \circ f(y)$ for all $x, y \in X$.

Note that if $f : X \to Y$ is a pseudo-BCI homomorphism, then $f(0_X) = 0_Y$, where $0_X$ and $0_Y$ are zero elements of $X$ and $Y$, respectively.

Let $f$ be a mapping from $X$ to $Y$. If $\mu$ is a fuzzy set in $X$, then the fuzzy set $f(\mu)$ in $Y$ defined by $f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in Y$, if $f^{-1}(y) = \emptyset$ then we put $f(\mu)(y) = 0$, is called the image of $\mu$ under $f$. Similarly, if $\nu$ is a fuzzy set in $Y$, then the fuzzy set $\mu = \nu \circ f$ in $X$, i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in X$ is called the preimage of $\nu$ under $f$.

Theorem 3.13. Any pseudo-BCI homomorphic preimage of a fuzzy pseudo-BCI ideal is also a fuzzy pseudo-BCI ideal.
Proof. Let \( f : X \to Y \) be a pseudo-BCI homomorphism of pseudo-BCI algebras. Let \( \nu \) be a fuzzy pseudo-BCI ideal of \( Y \) and let \( \mu \) be the preimage of \( \nu \) under \( f \). Then \( \nu(0_Y) \geq \nu(f(x)) = \nu(f(0_X)) = \mu(0_X) \), and so \( \mu(0_X) \geq \mu(x) \) for all \( x \in X \). Now we have

\[
\mu(x) \geq \nu(f(x)) \quad \forall x \in X \\
\geq \min\{\nu(f(x+y), \nu(f(y)) \quad \forall x, y \in Y \\
= \min\{\mu(x+y), \mu(y) \} \quad \forall x, y \in X,
\]

and similarly

\[
\mu(a) \geq \min\{\mu(a \cdot b), \mu(b) \} \quad \forall a, b \in X.
\]

Hence, by Theorem 3.3, \( \mu \) is a fuzzy pseudo-BCI ideal of \( X \).

Lemma 3.14. [3] Let \( f \) be a mapping from a set \( X \) to a set \( Y \) and let \( \mu \) be a fuzzy set in \( X \). Then \( U(f(\mu), t) = \bigcap_{0<s<t} f(U(\mu; t-s)) \) for all \( t \in (0,1] \).

Lemma 3.15. [5] Let \( f : X \to Y \) be an onto pseudo-BCI homomorphism of pseudo-BCI algebras. If \( I \) is a pseudo-BCI ideal of \( X \), then \( f(I) \) is a pseudo-BCI ideal of \( Y \).

By Lemma 3.14, Lemma 3.15, it is easy to show that

Theorem 3.16. An onto pseudo-BCI homomorphic image of a fuzzy pseudo-BCI ideal is a fuzzy pseudo-BCI ideal.

References


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