FUZZIFICATIONS OF PSEUDO-BCI IDEALS

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ABSTRACT. The fuzzification of pseudo-BCI ideals is considered, and some of their properties are investigated. Characterizations of pseudo-BCI ideals are provided. Also the homomorphic image and preimage of pseudo-BCI ideals are discussed.

1. INTRODUCTION

Georgescu and Iorgulescu [1] introduced the notion of a pseudo-BCK algebra as an extended notion of BCK-algebras. In [4], Jun, one of the present authors, gave a characterization of pseudo-BCK algebra, and provided conditions for a pseudo-BCK algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered). Jun et al. [7] introduced the notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra, and then they investigated some of their properties. In [2], Dudek and Jun introduced the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigated some properties. Jun et al. [5] introduced the concepts of pseudo-atoms, pseudo-BCI ideals and pseudo-BCI homomorphisms in pseudo-BCI algebras. They displaied characterizations of a pseudo-BCI ideal, and provided conditions for a subset to be a pseudo-BCI ideal. They also introduced the notion of a $\diamond$-medial pseudo-BCI algebra, and gave its characterization. In this paper, we consider the fuzzification of pseudo-BCI ideals, and investigate some of their properties. We give characterizations of pseudo-BCI ideals. We also discuss the homomorphic image and preimage of pseudo-BCI ideals.

2. PRELIMINARIES

Definition 2.1. [2] A *pseudo-BCI algebra* is a structure $\mathfrak{X} = (X, \preceq, *, \diamond, 0)$, where "$\preceq$" is a binary relation on a set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

(a1) $(x * y) \diamond (x * z) \preceq z * y$, $(x \diamond y) * (x \diamond z) \preceq z \diamond y$,
(a2) $x * (x \diamond y) \preceq y$, $x \diamond (x * y) \preceq y$,
(a3) $x \preceq x$,
(a4) $x \preceq y$, $y \preceq x \Longrightarrow x = y$,
(a5) $x \preceq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

Note that every pseudo-BCI algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$ is a BCI-algebra. Every pseudo-BCK algebra is a pseudo-BCI algebra.

Proposition 2.2. [2] *In a pseudo-BCI algebra $\mathfrak{X}$ the following holds:*

(p1) $x \preceq 0 \Rightarrow x = 0$.
(p2) $x \preceq y \Rightarrow z * y \preceq z * x$, $z \diamond y \preceq z \diamond x$.
(p3) $x \preceq y$, $y \preceq z \Rightarrow x \preceq z$.

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(p4) $(x * y) \diamond z = (x \diamond z) * y$.
(p5) $x * y \preceq z \Leftrightarrow x \diamond z \preceq y$.
(p6) $(x * y) * (z * y) \preceq x * z,\ (x \diamond y) \diamond (z \diamond y) \preceq x \diamond z$.
(p7) $x \preceq y \Rightarrow x * z \preceq y * z,\ x \diamond z \preceq y \diamond z$.
(p8) $x * 0 = x = x \diamond 0$.
(p9) $x * \big(x \diamond (x * y)\big) = x * y$ *and* $x \diamond \big(x * (x \diamond y)\big) = x \diamond y$.

3. Fuzzy Pseudo-BCI ideals

In what follows let $\mathfrak{X}$ denote a pseudo-BCI algebra unless otherwise specified. For any nonempty subset J of X and any element y of X, we denote

$$*(y, J) := \{x \in X \mid x * y \in J\} \text{ and } \diamond (y, J) := \{x \in X \mid x \diamond y \in J\}.$$

Note that $*(y, J) \cap \diamond(y, J) = \{x \in X \mid x * y \in J,\, x \diamond y \in J\}$.

Definition 3.1. [5] A nonempty subset J of $\mathfrak{X}$ is called a *pseudo-BCI ideal* of $\mathfrak{X}$ if it satisfies

(I1) $0 \in J$,
(I2) $\forall y \in J,\ *(y, J) \subseteq J$ and $\diamond (y, J) \subseteq J$.

Note that if $\mathfrak{X}$ is a pseudo-BCI algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notion of a pseudo-BCI ideal and a BCI-ideal coincide.

Definition 3.2. A fuzzy set $\mu : \mathfrak{X} \to [0, 1]$ is called a *fuzzy pseudo-BCI ideal* of $\mathfrak{X}$ if for every $t \in \mathrm{Im}(\mu)$, $U(\mu; t)$ is a pseudo-BCI ideal of $\mathfrak{X}$.

Theorem 3.3. *A fuzzy set $\mu : \mathfrak{X} \to [0, 1]$ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$ if and only if it satisfies:*

(i) $\mu(0) \geq \mu(x),\ \forall x \in X$.
(ii) $\mu(x) \geq \min\{\mu(x * y),\ \mu(y)\},\ \forall x, y \in X$.
(iii) $\mu(a) \geq \min\{\mu(a \diamond b),\ \mu(b)\},\ \forall a, b \in X$.

Proof. Assume that μ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$ and suppose that there exists $x_0 \in X$ such that $\mu(0) < \mu(x_0)$. Taking $t_0 := \frac{1}{2}(\mu(0) + \mu(x_0))$, we get $\mu(0) < t_0 < \mu(x_0)$. Hence $0 \notin U(\mu; t_0)$, which is a contradiction. Therefore $\mu(0) \geq \mu(x)$ for all $x \in X$. Assume that (ii) is false. Then

$$\mu(x_1) < \min\{\mu(x_1 * y_1),\ \mu(y_1)\}$$

for some $x_1, y_1 \in X$. Putting

$$t_1 := \frac{1}{2}(\mu(x_1) + \min\{\mu(x_1 * y_1),\ \mu(y_1)\}),$$

we have $\mu(x_1) < t_1 < \min\{\mu(x_1 * y_1),\ \mu(y_1)\}$. It follows that $x_1 * y_1 \in U(\mu; t_1)$ and $y_1 \in U(\mu; t_1)$, but $x_1 \notin U(\mu; t_1)$. This is a contradiction. Hence (ii) is true.
Simliarly we have (iii). □

Corollary 3.4. *Let μ be a fuzzy pseudo-BCI ideal of $\mathfrak{X}$. If $x \preceq y$ in $\mathfrak{X}$, then $\mu(x) \geq \mu(y)$.*

Proof. The proof is straightforward. □

We give conditions for a fuzzy set in $\mathfrak{X}$ to be a fuzzy pseudo-BCI ideal of $\mathfrak{X}$.

Theorem 3.5. *Let μ be a fuzzy set in $\mathfrak{X}$ such that*

(i) $\mu(0) \geq \mu(x),\ \forall x \in X$.
(ii) $\mu(x \diamond y) \geq \min\{\mu(((x * y) \diamond y) * z),\ \mu(z)\},\ \forall x, y, z \in X$.
(iii) $\mu(a * b) \geq \min\{\mu(((a \diamond b) * b) \diamond c),\ \mu(c)\},\ \forall a, b, c \in X$.

Then μ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$.

Proof. Taking $y = 0$ in (ii) and using (p8), we know that

$$\mu(x) \geq \min\{\mu(x * z), \mu(z)\}$$

for all $x, z \in X$. If we take $b = 0$ in (iii) and use the condition (p8), then

$$\mu(a) \geq \min\{\mu(a \diamond c), \mu(c)\}$$

for all $a, c \in X$. Hence, by Theorem 3.3, we have that μ is a fuzzy pseudo-ideal of $\mathfrak{X}$. □

Proposition 3.6. *If $\mu : \mathfrak{X} \to [0, 1]$ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$, then*

(i) $\forall x, y, z \in X,\ z * y \preceq x \ \Rightarrow\ \mu(z) \geq \min\{\mu(x), \mu(y)\}$.
(ii) $\forall a, b, c \in X,\ c \diamond b \preceq a \ \Rightarrow\ \mu(c) \geq \min\{\mu(a), \mu(b)\}$.

Proof. (i) Let $x, y, z \in X$ be such that $z * y \preceq x$. Then $\mu(z * y) \geq \mu(x)$ by Corollary 3.4, and so

$$\mu(z) \geq \min\{\mu(z * y), \mu(y)\} \geq \min\{\mu(x), \mu(y)\}$$

by Theorem 3.3(i). Now let $a, b, c \in X$ be such that $c \diamond b \preceq a$. Using Corollary 3.4, we get $\mu(c \diamond b) \geq \mu(a)$. It follows from Theorem 3.3(ii) that

$$\mu(c) \geq \min\{\mu(c \diamond b), \mu(b)\} \geq \min\{\mu(a), \mu(b)\}.$$

This completes the proof. □

The following is obvious from Theorem 3.3.

Theorem 3.7. *Let $\mu : \mathfrak{X} \to [0, 1]$ be a fuzzy set satisfying the following properties*

(i) $\mu(0) \geq \mu(x),\ \forall x \in X$.
(ii) $\forall x, y, z \in X,\ x * y \preceq z \ \Rightarrow\ \mu(x) \geq \min\{\mu(y), \mu(z)\}$.
(iii) $\forall a, b, c \in X,\ a \diamond b \preceq c \ \Rightarrow\ \mu(a) \geq \min\{\mu(b), \mu(c)\}$.

Then μ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$.

Theorem 3.8. *If μ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$, then the set*

$$I := \{x \in X \mid \mu(x) = \mu(0)\}$$

is a pseudo-BCI ideal of $\mathfrak{X}$.

Proof. Obviously, $0 \in I$. For any $y \in I$, let $x \in *(y, I)$ and $a \in \diamond(y, I)$. Then $x * y \in I$ and $a \diamond y \in I$, which imply that $\mu(x * y) = \mu(0) = \mu(a \diamond y)$. It follows from Theorem 3.3 that

$$\mu(x) \geq \min\{\mu(x * y), \mu(0)\} = \mu(0) \ \text{ and } \ \mu(a) \geq \min\{\mu(a * y), \mu(0)\} = \mu(0)$$

so from Theorem 3.3(i) that $\mu(x) = \mu(0) = \mu(a)$. Hence $x, a \in I$, which shows that $*(y, I) \subseteq I$ and $\diamond(y, I) \subseteq I$. This completes the proof. □

Definition 3.9. A fuzzy set $\mu : \mathfrak{X} \to [0, 1]$ is called a *fuzzy pseudo-BCI subalgebra* of $\mathfrak{X}$ if it satisfies

(i) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.
(ii) $\mu(a \diamond b) \geq \min\{\mu(a), \mu(b)\}$ for all $a, b \in X$.

It follows from the transfer principle ([6]) that

Theorem 3.10. *A fuzzy set $\mu : \mathfrak{X} \to [0, 1]$ is a fuzzy pseudo-BCI subalgebra of $\mathfrak{X}$ if and only if the nonempty level set $U(\mu; t)$ is a pseudo-BCI subalgebra of $\mathfrak{X}$ where $t \in \mathrm{Im}(\mu)$.*

Theorem 3.11. *For a fuzzy set μ in $\mathfrak{X}$, let $\mu^{\vee}$ be a fuzzy set in $\mathfrak{X}$ defined by*

$$\mu^{\vee}(x) := \sup\{t \in [0,1] \mid x \in \langle U(\mu;t)\rangle\},\ \forall x \in X.$$

Then $\mu^{\vee}$ is the least fuzzy pseudo-BCI ideal of $\mathfrak{X}$ that contains μ, where $\langle U(\mu;t)\rangle$ means the least pseudo-BCI ideal of $\mathfrak{X}$ containing $U(\mu;t)$.

Proof. At first we shall show

$$U(\mu^{\vee};t) = \bigcap_{\epsilon>0}\langle U(\mu;t-\epsilon)\rangle.$$

Let $x \in U(\mu^{\vee};t)$. Since $\mu^{\vee}(x) \geq t$, for every $\epsilon > 0$ there exists t^* such that

$$x \in U(\mu^{\vee};t^*) \quad \text{and} \quad t^* > t - \epsilon.$$

We have $U(\mu^{\vee};t^*) \subseteq U(\mu;t-\epsilon)$ by $t^* > t-\epsilon$. It follows that for every $\epsilon > 0$

$$x \in U(\mu;t-\epsilon) \subseteq \langle U(\mu;t-\epsilon)\rangle$$

and hence that

$$x \in \bigcap_{\epsilon>0}\langle U(\mu;t-\epsilon)\rangle.$$

Conversely, for every $x \in X$, we have

$$\begin{aligned} x \in \bigcap_{\epsilon>0}\langle U(\mu;t-\epsilon)\rangle &\Longrightarrow x \in \langle U(\mu;t-\epsilon)\rangle \text{ for every } \epsilon > 0 \\ &\Longrightarrow t-\epsilon \in \{t \in [0,1] \mid x \in \langle U(\mu;t)\rangle\} \text{ for every } \epsilon > 0 \\ &\Longrightarrow t-\epsilon \leq \sup\{t \in [0,1] \mid x \in \langle U(\mu;t)\rangle\} = \mu^{\vee}(x) \text{ for every } \epsilon > 0 \\ &\Longrightarrow t \leq \mu^{\vee}(x) \\ &\Longrightarrow x \in U(\mu^{\vee};t) \end{aligned}$$

Hence we have $U(\mu^{\vee};t) = \bigcap_{\epsilon>0}\langle U(\mu;t-\epsilon)\rangle$.

It is obvious that

$$U(\mu;t) = \bigcap_{\epsilon>0} U(\mu;t-\epsilon).$$

Since $\langle U(\mu;t-\epsilon)\rangle$ is a pseudo-BCI ideal and contains $U(\mu;t)$, we can conclude that $U(\mu^{\vee};t) = \bigcap_{\epsilon>0}\langle U(\mu;t-\epsilon)\rangle$ is the least pseudo-BCI ideal containing $U(\mu;t)$. It follows from transfer principle ([6]) that $\mu^{\vee}$ is the least fuzzy pseudo-BCI ideal of $\mathfrak{X}$ containing μ. □

Definition 3.12. [5] Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCI algebras. A mapping $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is called a *pseudo-BCI homomorphism* if $\mathfrak{f}(x * y) = \mathfrak{f}(x) * \mathfrak{f}(y)$ and $\mathfrak{f}(x \diamond y) = \mathfrak{f}(x) \diamond \mathfrak{f}(y)$ for all $x, y \in X$.

Note that if $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a pseudo-BCI homomorphism, then $\mathfrak{f}(0_{\mathfrak{X}}) = 0_{\mathfrak{Y}}$ where $0_{\mathfrak{X}}$ and $0_{\mathfrak{Y}}$ are zero elements of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively.

Let $\mathfrak{f}$ be a mapping from $\mathfrak{X}$ to $\mathfrak{Y}$. If μ is a fuzzy set in $\mathfrak{X}$, then the fuzzy set $\mathfrak{f}(\mu)$ in $\mathfrak{Y}$ defined by $\mathfrak{f}(\mu)(y) = \sup\limits_{x \in \mathfrak{f}^{-1}(y)} \mu(x)$ for all $y \in \mathfrak{Y}$, if $\mathfrak{f}^{-1}(y) = \emptyset$ then we put $\mathfrak{f}(\mu)(y) = 0$, is called the *image* of μ under $\mathfrak{f}$. Similarly, if ν is a fuzzy set in $\mathfrak{Y}$, then the fuzzy set $\mu = \nu \circ f$ in $\mathfrak{X}$, i.e., the fuzzy set defined by $\mu(x) = \nu(\mathfrak{f}(x))$ for all $x \in X$ is called the *preimage* of ν under $\mathfrak{f}$.

Theorem 3.13. *Any pseudo-BCI homomorphic preimage of a fuzzy pseudo-BCI ideal is also a fuzzy pseudo-BCI ideal.*

Proof. Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-BCI homomorphism of pseudo-BCI algebras. Let ν be a fuzzy pseudo-BCI ideal of $\mathfrak{Y}$ and let μ be the preimage of ν under $\mathfrak{f}$. Then $\nu(0_{\mathfrak{Y}}) \geq \nu(\mathfrak{f}(x)) = \mu(x)$ for all $x \in X$. But $\nu(0_{\mathfrak{Y}}) = \nu(\mathfrak{f}(0_{\mathfrak{X}})) = \mu(0_{\mathfrak{X}})$, and so $\mu(0_{\mathfrak{X}}) \geq \mu(x)$ for all $x \in X$. Now we have

$$\begin{aligned} \mu(x) &\geq \nu(\mathfrak{f}(x)) \ \forall x \in X \\ &\geq \min\{\nu(\mathfrak{f}(x * y) * \nu(\mathfrak{f}(y)\} \ \forall x, y \in Y \\ &= \min\{\mu(x * y), \mu(y)\} \ \forall x, y \in X, \end{aligned}$$

and similarly

$$\mu(a) \geq \min\{\mu(a \diamond b), \mu(b)\} \ \forall a, b \in X.$$

Hence, by Theorem 3.3, μ is a fuzzy pseudo-BCI ideal of $\mathfrak{X}$. □

Lemma 3.14. [3] *Let $\mathfrak{f}$ be a mapping from a set X to a set Y and let μ be a fuzzy set in X. Then $U(\mathfrak{f}(\mu), t) = \bigcap_{0<s<t} \mathfrak{f}(U(\mu; t - s))$ for all $t \in (0, 1]$.*

Lemma 3.15. [5] *Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be an onto pseudo-BCI homomorphism of pseudo-BCI algebras. If I is a pseudo-BCI ideal of $\mathfrak{X}$, then $\mathfrak{f}(I)$ is a pseudo-BCI ideal of $\mathfrak{Y}$.*

By Lemma 3.14, Lemma 3.15, it is easy to show that

Theorem 3.16. *An onto pseudo-BCI homomorphic image of a fuzzy pseudo-BCI ideal is a fuzzy pseudo-BCI ideal.*

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