NOTES ON INTERPOLATION THEOREM BETWEEN B^p AND BMO

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Abstract. We show that the sharp function $f^\sharp$ belongs to B^p (the dual of Beurling algebra) if and only if f belongs to CMO^p (central mean oscillation). We also show the interpolation theorem between B^p and BMO.

1 Introduction The John-Nirenberg space BMO is characterized by the sharp function $f^\sharp$ (see Section 2). Matsuoka [4] obtained the estimates of $f^\sharp$ on B^p and CMO^p. In this paper, we refine his results and show that f is in CMO^p if and only if $f^\sharp$ is in B^p. Applying our theorem, we obtain the interpolation theorem between B^p and BMO. Our theorem is applicable to Calderón-Zygmund operators.

2 Definitions The following notation is used: For a set $E \subset R^n$ we denote the characteristic function of E by χ_E and $|E|$ is the Lebesgue measure of E.

We denote a ball of radius R centered at origin by $B(0,R)$ and for any ball Q, we denote the radius of Q by $\mathrm{rad}(Q)$.

First we define some function spaces which we shall consider in this paper.

Definition 1. Let $1 < p < \infty$ and let

$$B^p = \left\{ f \in L^p_{loc}(R^n) : \|f\|_{B^p} = \sup_{R \ge 1} \Big(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x)|^p dx \Big)^{1/p} < \infty \right\}.$$

Definition 2. Let $1 < p < \infty$ and let

$$CMO^p = \Big\{ f \in L^p_{loc}(R^n) : \|f\|_{CMO^p} = \sup_{R \ge 1} \Big(\frac{1}{|B(0,R)|} \int_{B(0,R)} |f(x) - m_R(f)|^p dx \Big)^{1/p} < \infty \Big\},$$

where

$$m_R(f) = \frac{1}{|B(0,R)|} \int_{B(0,R)} f(x) dx.$$

Remark . $B^p \subset CMO^p$.

About basic properties of B^p and CMO^p, see for example [1] and [2].

Definition 3. We define

$$BMO = \Big\{ f \in L^1_{loc}(R^n) : \|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty \Big\},$$

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where the supremum is taken over all balls $Q \subset R^n$ and

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx.$$

Definition 4. For any $f \in L^1_{loc}$, the sharp function $f^\sharp$ is defined by

$$f^\sharp(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

where the supremum is taken over all balls Q containing x.

Remark . $BMO \subset CMO^p$ and $\|f^\sharp\|_{L^\infty} = \|f\|_{BMO}$.

3 Results Matsuoka [4] obtained the following:

Theorem A . *Let $1 < p < \infty$. Then*

$$\|f^\sharp\|_{B^p} \le C_p\|f\|_{B^p},$$

where C_p is a positive constant depending only on p and n.

Theorem B . *Let $1 < p < \infty$. Then*

$$\|f\|_{CMO^p} \le C_p\|f^\sharp\|_{B^p},$$

if

$$\lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(0,R)} |f(x)|^p dx = 0.$$

Our results are the following:

Theorem 1. *Let $1 < p < \infty$. Then*

$$\|f^\sharp\|_{B^p} \le C_p\|f\|_{CMO^p}.$$

Theorem 2. *Let $1 < p < \infty$. If $f \in L^p_{loc}$, then*

$$\|f\|_{CMO^p} \le C_p\|f^\sharp\|_{B^p}.$$

Applying our theorems, we obtain the next interpolation theorem. Matsuoka [3], [4] proved the following:

Theorem C . *Suppose $1 < p_0 < \infty$, and let T be a sublinear operator such that*

$$\begin{aligned} &T \text{ is bounded from } B^{p_0} \text{ to } B^{p_0} \quad \text{and} \\ &T \text{ is bounded from } L^\infty \text{ to } L^\infty. \end{aligned}$$

Then T is bounded from B^p to B^p where $p_0 < p < \infty$.

Our result is the following:

Theorem 3. *Suppose $1 < p_0 < \infty$, and let T be a sublinear operator such that*

$$(*) \qquad \begin{aligned} &T \text{ is bounded from } B^{p_0} \text{ to } CMO^{p_0} \quad \text{and} \\ &T \text{ is bounded from } L^\infty \text{ to } BMO. \end{aligned}$$

Then T is bounded from B^p to CMO^p where $p_0 < p < \infty$.

Remark . The condition $(*)$ is natural, because Calderón-Zygmund operators satisfy $(*)$ (see [2]).

4 Lemmas To prove our theorems we need some lemmas. First we show the boundedness of the maximal function.

Definition 5. The Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all balls Q containing x.

Lemma 1 ([1], [2]). *Let $1 < p < \infty$. Then*

$$\|Mf\|_{B^p} \le C_p \|f\|_{B^p}.$$

Next we define Beurling algebra A^p.

Definition 6. Let $1 < p < \infty$ and let

$$A^p = \Big\{ f : \|f\|_{A^p} = \|f \cdot \chi_{B(0,1)}\|_{L^p} + \sum_{k=1}^{\infty} 2^{kn/p'} \|f \cdot \chi_{B(0,2^k)\setminus B(0,2^{k-1})}\|_{L^p} < \infty \Big\},$$

where $1/p + 1/p' = 1$.

Next we define atom and some Hardy space.

Definition 7. Let $1 < p < \infty$. We say a function $a(x)$ is a central $(1,p)$-atom if a satisfies the following:

$$\begin{aligned} &\mathrm{supp}(a) \subset B(0,R) \quad \text{for some} \quad R \ge 1, \\ &\Big(\frac{1}{|B(0,R)|} \int_{B(0,R)} |a(x)|^p dx\Big)^{1/p} \le \frac{1}{|B(0,R)|}, \\ &\int a(x)dx = 0. \end{aligned}$$

Definition 8. Let $1 < p < \infty$. We define the Hardy space HA^p by

$$HA^p = \Big\{ f : f(x) = \sum_{j=1}^{\infty} c_j a_j(x),\ a_j\text{'s are central } (1,p)\text{-atoms and} \sum_{j=1}^{\infty} |c_j| < \infty \Big\},$$

and the norm $\|f\|_{HA^p}$ be the infimum of $\sum_{j=1}^{\infty} |c_j|$ over all representations of f.

Chen and Lau [1] and García-Cuerva [2] obtained the following duality theorems.

Lemma 2.

$$(A^p)^* = B^{p'}, \quad \text{where} \quad 1/p + 1/p' = 1.$$

Lemma 3.

$$(HA^p)^* = CMO^{p'}, \quad \text{where} \quad 1/p + 1/p' = 1.$$

HA^p is characterized by the grand maximal function.

Lemma 4. *If $f \in HA^p$ then $\mathcal{M}f \in A^p$ and $\|f\|_{HA^p} \approx \|\mathcal{M}f\|_{A^p}$.*

See [2] for the definition of $\mathcal{M}$ and the proof of this lemma.

The next lemma is trivial.

Lemma 5.

$$\|f\|_{B^p} \approx \Big(\frac{1}{|B(0,4)|}\int_{B(0,4)} |f(x)|^p dx\Big)^{1/p} + \sup_{k\geq 3}\Big(\frac{1}{2^{kn}}\int_{B(0,2^k)\setminus B(0,2^{k-1})} |f(x)|^p dx\Big)^{1/p}.$$

The following lemma is proved in [5], p. 146.

Lemma 6. *Suppose that $f \in L^p_{loc}$ and a is a central $(1,p')$-atom. Then*

$$\Big|\int f(x)a(x)dx\Big| \leq C\int f^\sharp(x)\mathcal{M}a(x)dx.$$

5 Proof of Theorem 1 By Lemma 5, it suffices to show the following two inequalities.

$$\text{(I)} \qquad \frac{1}{|B(0,4)|}\int_{B(0,4)} f^\sharp(x)^p dx \leq C_p\|f\|^p_{CMO^p},$$

$$\text{(II)} \qquad \frac{1}{2^{kn}}\int_{B(0,2^k)\setminus B(0,2^{k-1})} f^\sharp(x)^p dx \leq C_p\|f\|^p_{CMO^p}, \quad \text{where} \quad k\geq 3.$$

We shall prove only (II). The proof of (I) is similar.

We write

$$\begin{aligned} f^\sharp(x) &\leq \sup_{x\in Q,\mathrm{rad}(Q)\geq 2^{k-3}} \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy + \sup_{x\in Q,\mathrm{rad}(Q)\leq 2^{k-3}} \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy \\ &= f^{\sharp(1)}(x) + f^{\sharp(2)}(x). \end{aligned}$$

First we estimate $f^{\sharp(1)}(x)$ on $B(0,2^k)\setminus B(0,2^{k-1})$.

If $x\in B(0,2^k)\setminus B(0,2^{k-1}), x\in Q$ and $\mathrm{rad}(Q)\geq 2^{k-3}$, then there exists a ball such that $Q\subset B(0,R)$ and $R\leq 10\cdot\mathrm{rad}(Q)$. So we have

$$\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy \leq \frac{2\cdot 10^n}{|B(0,R)|}\int_{B(0,R)} |f(y)-m_R(f)|dy,$$

and we obtain $f^{\sharp(1)}(x) \leq C_n\|f\|_{CMO^p}$.

Next we estimate $f^{\sharp(2)}(x)$ on $B(0,2^k)\setminus B(0,2^{k-1})$.

We define

$$g(x) = (f(x) - m_{2^{k+1}}(f))\cdot\chi_{B(0,2^{k+1})\setminus B(0,2^{k-2})}(x).$$

Suppose that $x\in B(0,2^k)\setminus B(0,2^{k-1}), x\in Q$ and $\mathrm{rad}(Q)\leq 2^{k-3}$. Then

$$f(y)-f_Q = g(y)-g_Q \quad \text{for all} \quad y\in Q.$$

So we have

$$f^{\sharp(2)}(x)\leq 2Mg(x).$$

By Lemma 1, we obtain

$$\begin{aligned} \frac{1}{2^{kn}}\int_{B(0,2^k)\setminus B(0,2^{k-1})} f^{\sharp(2)}(x)^p dx &\leq C_p\|Mg\|^p_{B^p} \leq C_p\|g\|^p_{B^p} \\ &\leq \frac{C_p}{2^{(k+1)n}}\int_{B(0,2^{k+1})} |f(x)-m_{2^{k+1}}(f)|^p dx \leq C_p\|f\|^p_{CMO^p}. \end{aligned}$$

6 Proof of Theorem 2 By the definition of HA^p and Lemma 3, it suffices to show

$$\left|\int f(x)a(x)dx\right| \leq C_p\|f^\sharp\|_{B^p} \quad \text{for any central } (1,p')\text{-atom.}$$

By Lemma 6, Lemma 2 and Lemma 4, we have

$$\left|\int f(x)a(x)dx\right| \leq C\int f^\sharp(x)\mathcal{M}a(x)dx \leq C_p\|f^\sharp\|_{B^p}\|\mathcal{M}a\|_{A^{p'}} \leq C_p\|f^\sharp\|_{B^p}.$$

7 Proof of Theorem 3 We have

$$\|(Tf)^\sharp\|_{B^{p_0}} \leq C_{p_0}\|Tf\|_{CMO^{p_0}} \leq \|f\|_{B^{p_0}}$$

by Theorem 1, and we have

$$\|(Tf)^\sharp\|_{L^\infty} = \|Tf\|_{BMO} \leq C\|f\|_{L^\infty}.$$

So we obtain

$$\|(Tf)^\sharp\|_{B^p} \leq C_p\|f\|_{B^p} \quad \text{where} \quad p_0 < p < \infty,$$

by Theorem C, and we have

$$\|Tf\|_{CMO^p} \leq C_p\|(Tf)^\sharp\|_{B^p} \leq C_p\|f\|_{B^p}$$

by Theorem 2.

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