An estimation of Hadamard product by sequential product of operators

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Abstract. For operators A and B on a Hilbert space, the sequential product is defined by $A \boxdot B = U|A|^{\frac{1}{2}}B|A|^{\frac{1}{2}}$, where $A = U|A|$ is the polar decomposition of A. In particular, $A \boxdot B = A^{\frac{1}{2}}BA^{\frac{1}{2}}$ for positive operators A and B, which coincides with Gudder-Nagy's definition for positive contractions. In this note, as a complement of a recent result by Hiramatsu and Seo, we give an estimation of Hadamard product $A \circ B$ by sequential product $A \boxdot B$: For positive invertible operators A and B with the condition numbers $h_A(= \|A\|\|A^{-1}\|)$ and h_B respectively,

$$\frac{1}{h_A h_B} A \boxdot B \le A \circ B \le h_A h_B A \boxdot B.$$

More generally, it is considered for any quasi-mean. As an application, we have an estimation by weighted operator fidelity $A \hat{\#}_t B = (A \boxdot B)^t$ as follows: If $m_A I \le A \le M_A I$ and $m_B I \le B \le M_B I$ for some $m_A, M_A, m_B, M_B > 0$, then

$$\frac{m_A m_B}{(M_A M_B)^t} A \hat{\#}_t B \le A \circ B \le \frac{M_A M_B}{(m_A m_B)^t} A \hat{\#}_t B.$$