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EXTENSIONS OF ANDO-HIAI INEQUALITY WITH NEGATIVE POWER

Dedicated to the 100th anniversary of the birth of
the late Professor Masahiro Nakamura

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Abstract. The Ando-Hiai inequality says that if $A\#_\alpha B \le 1$ for a fixed $\alpha \in [0,1]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\alpha B^r \le 1$ for $r \ge 1$, where $\#_\alpha$ is the α-geometric mean defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. In this note, we generalize it as follows: If $A\natural_\alpha B \le 1$ for a fixed $\alpha \in [-1,0]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\beta B^s \le 1$ for $r \in [0,1]$ and $s \in [\frac{-2\alpha r}{-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$ and $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. As an application, we pose operator inequalities of type of Furuta inequality and grand Furuta inequality. For instance, if $A \ge B > 0$, then $A^{-r}\natural_{\frac{1+r}{p+r}}B^p \le A$ holds for $p \le -1$ and $r \in [-1,0]$, where $A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

Key words and phrases. Ando-Hiai inequality, generalized Ando-Hiai inequality, Furuta inequality, grand Furuta inequality, operator geometric mean .